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Recover Data in Sparse Expansion Forms Modeled by Special Basis Functions

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Abstract

In data analysis and signal processing, the recovery of structured functions (in terms of frequencies and coefficients) with respect to certain basis functions from the given sampling values is a fundamental problem. The original Prony method is the main tool to solve this problem, which requires the equispaced sampling values.

In this dissertation, we use the equispaced sampling values in the frequency domain after the short time Fourier transform in order to reconstruct some signal expansions, such as the exponential expansions and the cosine expansions. In particular, we consider the case that the phase of the cosine expansion is quadratic. Moreover, we work on the expansion problem based on the eigenfunctions of some linear operators. In addition, when the signals contain two different models, we develop a method that separate the signals in single-models and then solve the problem. We also consider the situation that when some of the sampling values are corrupted.

Keywords: Prony Method, Exponential Sums, Eigenfunctions, Eigenvalues, B-Spline, Fourier Transform, Short Time Fourier Transform, Frequency Domain, Resultant, Sylvester Matrix.

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Contents

1 Introduction **10**

2 Preliminaries **13**
2.1 Fourier Series 13
2.2 Fourier Transform (FT) 14
2.3 Short-Time Fourier Transform (STFT) 16
2.4 B-Splines 16
2.5 Hankel,Vandermonde and Toeplitz Matrices 17
2.6 Generalized Shift Operators 18
2.7 Gaussian Integral 19
2.8 Chebyshev Polynomials 19
2.9 Resultant and the Sylvester Matrix 20

3 Prony Method and its Variations **21**
3.1 Prony method with $2M$ sample values 21
3.2 Prony method with $M+1$ sample values 25
3.3 Prony method for sparse expansion of eigenfunctions 28
3.4 Non Uniform Spline Expansion 31
3.4.1 Reconstruction of Characteristic Expansion 32
3.4.2 Reconstruction of Non Uniform Spline Expansion 33
3.5 Reconstruction of Cosine Expansions Using the Chebyshev Polynomial 34
3.6 Reconstruction of Expansion of shift Gaussian 36

4 Generalization of Prony Method **39**
4.1 Reconstruction of Exponential Sums Using Short Time Fourier Transform 39
4.2 Reconstruction of Mixed Expansions Using Generalized Shifted Operator 41
4.3 Reconstruction of Cosine Expansion Using Short Time Fourier Sample 46
4.4 Reconstruction of Non-stationary Signals with Quadratic Phase Functions Using Short Time Fourier Transform 49
4.5 Reconstruction of Expansions Using Variable Substitutions 52

4.6 Reconstruction of Expansions Using Polynomials of Differential Operators . . . 56
4.6.1 Prony Method for Exponential Sums Using the Polynomial of Differential Operators . . . 57
4.6.2 Reconstruction of Cosine Expansions Using Polynomials of Differential Operators . . . 58
4.7 Reconstruction of Gaussian Expansions Using Differential Operators 61
4.8 Reconstruction Expansions Using Combination of Shift and Dilation Operators . . . 62
4.9 Numerical Experiments . . . 64

5 Expansions with Two Models **70**
5.1 Reconstruction of Linear Combinations of Sine and Cosine Expansions Using the Polynomials of Differential Operators . . . 70
5.2 Reconstruction of Linear Combinations of Sine and Cosine Expansions Using Chebyshev Polynomials . . . 73
5.3 Reconstruction of Expansions of Linear Combinations of Two Models Using Odd and Even Properties . . . 77
5.4 Numerical Experiments . . . 78

6 Fixing Corrupted Sampling Values **81**
6.1 Reduction Method . . . 81
6.2 Determinant Method . . . 87
6.3 Resultant Method . . . 89
6.4 Repeated Resultant Method . . . 92

7 Summary and Future Work **96**

List of Figures

3.1 The blue line shows the function $f(x)$ in (3.2.9) while 6 red stars are the sampling values. 22

4.1 The signal $f(x)$ in (4.9.1) and the three components $f_1(x)$, $f_2(x)$, and $f_3(x)$. 66
4.2 Components of the signal $f(x)$ in (4.9.1) using short time Fourier sampling values. 67
4.3 The signal $f(x)$ in (4.9.3) and the three components $f_1(x)$, $f_2(x)$, and $f_3(x)$. 68
4.4 Components of the signal $f(x)$ in (4.9.3) using short time Fourier sampling values. 68

5.1 The signal $f(x)$ in (5.4.2) with 8 sampling values. 79

List of Tables

3.1 Parameters of the function $f(x)$ in (3.2.9) and approximate errors with $h = 0.5$ 28

4.1 Parameters of the function $f(x)$ in (4.9.1) 65
4.2 Parameters of the function $f(x)$ in (4.9.1) and approximate errors using 6 short time Fourier sampling values with $h = 0.5$ 65
4.3 Parameters of the function $f(x)$ in (4.9.3). 66
4.4 Parameters of the function $f(x)$ in (4.9.3) and approximate errors using 12 sampling values with $h = 0.5$. 67
4.5 Parameters of the function $f(x)$ in (4.9.3) and approximate errors using 12 sampling values as in (4.6.14). 67
4.6 Parameters of the function $f(x)$ in (4.9.5) 68
4.7 Parameters of the function $f(x)$ in (4.9.5) and approximate errors using 10 sampling values with $h = 0.5$. 69
4.8 Parameters of the function $f(x)$ in (4.9.6). 69
4.9 Parameters of the function $f(x)$ in (4.9.6) and approximate errors using 12 sampling values with $h = 3$. 69

5.1 Parameters of the function $f(x)$ in (5.4.1) 78
5.2 Parameters of the function $f(x)$ in (5.4.1) and approximate errors using 6 sampling values with $x_0 = 1$ and $P_n(x) = 2x^2 + 3x - 1$. . . 79
5.3 Parameters of the function $f(x)$ in (5.4.2) 79
5.4 Approximate errors using 8 sampling values with $x_0 = 1$ and $h = 0.4$ of the function (5.4.2). 80

Abbreviations

FT: Fourier Transform.
STFT: Short time Fourier Transform.
H_N: Hankel matrix.
T_N:Toeplitz matrix.
V_N: Vandermonde matrix.
Res: Resultant.
Syl: Sylvester matrix.

Chapter 1

Introduction

In applied mathematics and signal processing, we frequently encounter this fundamental problem: Given a set of sampling data points, how to choose suitable modelling functions such that the data can be represented as an expansion of the modelling functions.

The solution of this problem relies on two critical factors: the modelling functions and the sampling data points. The first model that was brought to people's attention by Gaspard Riche de Prony was the exponential model. To recover an exponential signal, we consider the following form

$$f(x) = \sum_{j=1}^{M} c_j e^{x\phi_j}, \tag{1.0.1}$$

where ϕ_j's and c_j's are the parameters to be determined from a set of function values. The well-known Prony method solves this problem with $2M$ equispaced sampling values $f(lh), l = 0, ..., 2M-1$ for some positive constant h. Since then the Prony method has been extended and generalized to solve many different signal models.

During the last few years, the Prony method is widely used in different fields, such as identification and spectral estimation [**20**], the approximation of Green functions [**20**]. Several generalizations of the original Prony method formulation [**4**, **12** , **13**, **14** , **15** , **16** , **17**, **18**, **26**, **30**] have been developed over the last few years. Peter and Plonka in [**12**, **15**] generalized the Prony method to reconstruct M-sparse expansions in term of eigenfunctions of some special linear operators. They showed that all well-known Prony reconstruction methods can be uniformly interpreted as special cases of their formulation.

A fundamental problem in signal processing is the *estimation of parameters*. Potts and Tasche in [**20**] have shown that ESPRIT method (estimation of signal parameters via rotational technique) [**21**], Matrix pencil method [**8**] and MUSIC method (multiple signal classification) [**22**] can be viewed as Prony-like methods.

The generalizations of the Prony method have been also studied to recover the parameters of Legendre expansion [**14**]. Moreover, Plonka and Marius in [**19**, **30**] used the techniques of Fourier transform in order to recover the non-uniform B-Spline expansion by using a small number of equidistantly sampling values. In [**17**] Plonka and others presented the reconstruction of different signals by exploiting the generalized shift operator.

In this dissertation, we explore several tools that can help us reconstruct the signal functions with various structures. The short time Fourier transform (STFT) is a prominent tool that helps us recover signals in several models from the short time Fourier sampling values. Moreover, the non-stationary signal with quadratic phase can also be reconstructed using the short time Fourier sampling. By knowing the reconstruction of $f(x) = \sum_{j=1}^{M} c_j x^{\phi_j}$ [**12**],[**17**], we connect it to several other more complex models through appropriate inverse functions, and then apply the variable substitution method to solve the problem. Another powerful tool that is effective in many situations is based on the differential operators. When the signals are represented as expansions of the eigenfunctions of some differential operators, this method works well. We also consider the signals represented in mixed models. For example, we study the signals with the following form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) + \sum_{k=1}^{N} d_j \sin(\beta_k x), \tag{1.0.2}$$

where c_j's, ϕ_j's, d_k's, and β_k's are the parameters to be determined from a set of function values. We separate the signals in the individual models using the even-odd properties of the functions, and then recover the parameters using the single-model methods. We also study the problem with *oversampled* data points. We consider the situation that there are some sampling values that could be *incorrect*. Since the Prony method has a property: If one sampling value changes slightly, then the computation result would change dramatically. With this observation, in order to detect and fix those corrupted sampling values, we develop a determinant-based method that allows us to recover the original signals.

This dissertation is organized as follow : Preliminaries in chapter 2 , which provide us of the concepts that we will use through this dissertation. In chapter 3, we introduce Prony method and we discuss some of its variation. In chapter 4, we introduce new generalizations of the method in order to reconstruct more signals types. In chapter 5, we present method that allow us to reconstruct signals that have two different models.In chapter 6 we present our method for using the Prony method to recover the signals even if the sampling values contain some incorrect values. Finally, we end this dissertation with a summary and future work.

Chapter 2

Preliminaries

2.1 Fourier Series

Given any periodic function $f(x)$ with a period T (a commonly used T is 2π), (i.e $f(x+T) = f(x)$ for any $x \in \mathbb{R}$), then this type of functions can be decomposed into an infinite series of sine and cosine functions, which is called the Fourier series of f and denoted by Sf, and it has the form:

$$(Sf) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\Big), \tag{2.1.1}$$

where

$$\begin{cases} a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{k\pi x}{L})dx, \quad k = 0, 1, ...; \\ \\ b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{k\pi x}{L})dx, \quad k = 1, 2, ... \end{cases} \tag{2.1.2}$$

are called the Fourier coefficients of the function f. From (2.1.1), we can see that $\left\{1, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)\right\}$ play a role of basis functions for the real-valued functions defined on an interval of length L. The following theorem gives a more precise description about this observation.

Theorem 2.1.1 (Fourier series). *Given $L > 0$, the family*

$$\left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi kx}{L}\right), \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi kx}{L}\right), \quad k = 1, 2, ... \right\} \tag{2.1.3}$$

is an orthonormal basis of $L_2[-\frac{L}{2}, \frac{L}{2}]$. Consequently, any $f \in L_2[-\frac{L}{2}, \frac{L}{2}]$ can be represented by its Fourier series, namely

$$f(x) = (Sf) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\Big), \tag{2.1.4}$$

which converges to f *in* $L_2[-\frac{L}{2}, \frac{L}{2}]$ *and* a_0,$a_k, b_k, (k = 1, 2, ...)$ *are the Fourier coefficients defined in* (2.1.2).

If we consider f as an even function on $[-\frac{L}{2}, \frac{L}{2}]$, then (2.1.2) tells us that all the b_k coefficients vanish, which means that all the sine terms in the Fourier series expansion of f disappear. With this observation, when we have a function on $[0, \frac{L}{2}]$, we can just do an even extension on this function, that is, define this function on $[-\frac{L}{2}, 0]$ as $f(x) = f(-x)$. In this way, we get an even function on $[-\frac{L}{2}, \frac{L}{2}]$. Thus we have the following version derived from theorem (2.1.1), which gives us an alternative way to process signals.

Theorem 2.1.2 (Fourier cosine series). *Given* $L > 0$*, the family*

$$\left\{\frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi kx}{L}\right), \quad k = 1, 2, ...\right\} \tag{2.1.5}$$

is an orthonormal basis of $L_2[0, \frac{L}{2}]$*. Consequently, any* $f \in L_2[0, \frac{L}{2}]$ *can be expanded by its Fourier cosine series, namely*

$$f(x) = (S^e f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right), \tag{2.1.6}$$

which converges to $f(x)$ *in* $L_2[0, \frac{L}{2}]$ *where* a_k $(k = 0, 1, ...)$ *are the Fourier cosine coefficients and given by*

$$a_k = \frac{4}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi kx}{L}\right)dx, \quad k = 0, 1, ... \tag{2.1.7}$$

2.2 Fourier Transform (FT)

The Fourier transform is one of the most important tools to analyse signals.The Fourier series discussed in the section (2.1) allow us to deal with periodic functions in order to study the frequency contents. In this section, we provide Fourier transform and some useful properties that allow us to deal with non-periodic functions.

Definition 2.2.1 (Fourier transform). Let L_1 be the space of all integrable functions and let f is a function in $L_1(\mathbb{R})$, then the Fourier transform of f denoted by $\hat{f}$ or $\mathbb{F}(f)$ is given by the integral

$$\hat{f}(\omega) = (\mathbb{F}f)(\omega) = \int_{\infty}^{-\infty} f(x)e^{-i2\pi\omega x}dx, \quad \omega \in \mathbb{R} \tag{2.2.1}$$

Next, we state some useful properties for the Fourier transform which provide us with some nice insights into the behaviour of the Fourier transform.

Definition 2.2.2. Given $a > 0$, $b, c \in \mathbb{R}$, and $f \in L_1(\mathbb{R})$ then

1. The dilation operator D_a is given by

$$(D_a f)(x) := f(ax) \tag{2.2.2}$$

2. The translation operator T_b is given by

$$(T_b f)(x) := f(x - b) \tag{2.2.3}$$

3. The (frequency) modulation operator M_c is given by

$$(M_c f)(x) := f(x)e^{i2\pi cx}, \quad c \neq 0 \tag{2.2.4}$$

Definition 2.2.3. Given two functions f, h on $\mathbb{R}$, The convolution of $f(x)$ and $h(x)$ denoted by $(f * h)(x)$ and is defined by

$$(f * h)(x) = \int_{-\infty}^{\infty} f(t)h(x - t)dt \tag{2.2.5}$$

Theorem 2.2.4 (Properties of Fourier transform). *The Fourier transform defined in* 2.2.1 *has the following properties:*

1. *Given D_a the dilation operator defined in* (2.2.2)*, then for $f \in L_1(\mathbb{R})$,*

$$\widehat{(D_a f)}(\omega) = \frac{1}{a}\hat{f}(\frac{\omega}{a}). \tag{2.2.6}$$

2. *Given T_a the translation operator defined in* (2.2.3)*, then for $f \in L_1(\mathbb{R})$,*

$$\widehat{(T_b f)}(\omega) = e^{-i2\pi\omega b}\hat{f}(\omega). \tag{2.2.7}$$

3. *Given M_c the (frequency) modulation operator defined in* (2.2.4)*, then for $f \in L_1(\mathbb{R})$,*

$$\widehat{(M_c f)}(\omega) = \hat{f}(\omega - c). \tag{2.2.8}$$

4. *Given $f, h \in L_1(\mathbb{R})$, then*

$$(\widehat{f * h})(\omega) = \hat{f}(\omega)\hat{h}(\omega). \tag{2.2.9}$$

2.3 Short-Time Fourier Transform (STFT)

Sometimes when we analyse signals, we are interested in certain local sections of the signals. To this end, we introduce a window function w that makes our interested portion of the signals stand out.

Definition 2.3.1. Given a window function $w \in (L_1 \cap L_2)(\mathbb{R})$ and $\tau \in \mathbb{R}$, then for every $f \in L_2(\mathbb{R})$, the *short time Fourier transform* of f is defined by the integral

$$STFT(f(x)) = \mathcal{F}(\omega, \tau) = \int_{-\infty}^{\infty} f(x)w(x-\tau)e^{-i\omega x}dx. \tag{2.3.1}$$

Short time Fourier transform preserves time shift up to modulation. Let us define the shifted signal as $g(x) = f(x-a)$, then its short time Fourier transform becomes

$$\begin{aligned} STFT(g(x)) = \mathcal{G}(\omega, \tau) &= \int_{-\infty}^{\infty} f(x-a)w(x-\tau)e^{-i\omega x}dx \\ &= \int_{-\infty}^{\infty} f(u)w(u-(\tau-a))e^{-i\omega u}e^{-i\omega a}du \\ &= e^{-i\omega a}\mathcal{G}(\omega, \tau-a). \end{aligned} \tag{2.3.2}$$

2.4 B-Splines

B-splines give us another powerful tool in data representation. A B-spline function is a piecewise polynomial that is defined on a sequence of knots.

Definition 2.4.1. Given a non-decreasing knot sequence $\{T_j, T_{j+1}, ..., T_{j+m}\}$ with $T_j < T_{j+m}$, then the m-th order B-spline basis functions can be defined by the following recurrence formula

$$N_j^m(x) = \frac{x - T_j}{T_{j+m-1} - T_j} N_j^{m-1}(x) + \frac{T_{j+m} - x}{T_{j+m} - T_{j+1}} N_{j+1}^{m-1}(x) \tag{2.4.1}$$

with

$$N_j^1(x) = \begin{cases} 1, & \text{if} \quad x \in [T_j, T_{j+1}); \\ 0, & \text{otherwise}, \end{cases} \tag{2.4.2}$$

where the fractions with zero denominator are assumed to be zero.

Theorem 2.4.2 (Properties of B-spline Basis Functions). *The B-spline basis functions defined in* (2.4.1) *and* (2.4.2) *have the following properties*

1. *Positivity,*
$$N_j^m(x) > 0, \quad \textit{for all} \quad x \in (T_j, T_{j+m}). \tag{2.4.3}$$

2. *Local support,*
$$N_j^m(x) = 0, \quad \textit{for all} \quad x \notin (T_j, T_{j+m}). \tag{2.4.4}$$

3. *Partition of Unity,*
$$\sum_{j=n+1-m}^{n} N_j^m(x) = 1, \quad \textit{for all} \quad x \in [T_n, T_{n+1}). \tag{2.4.5}$$

4. *Smoothness at a knot,*
$$N_j^m(x) \in \mathcal{C}^{m-k_i}(T_i), \quad j = 1, ..., n, \tag{2.4.6}$$
where k_i is the multiplicity of T_i.

5. *The derivative of B-spline basis functions is*
$$\frac{d}{dx}\left(N_j^m(x)\right) = \frac{m-1}{T_{j+m-1} - T_j} N_j^{m-1}(x) - \frac{m-1}{T_{j+m} - T_{j+1}} N_{j+1}^{m-1}(x). \tag{2.4.7}$$

2.5 Hankel,Vandermonde and Toeplitz Matrices

There are some special types of matrices with special properties that are very useful in solving certain types of problems. In this section, we introduce three of them.

Definition 2.5.1. Given a sequence of real numbers $S = \{\alpha_1, \alpha_2, ..., \alpha_{2n-1}\}$, the *Hankel matrix* is an $n \times n$ matrix with the following structure
$$H = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_2 & \alpha_3 & \dots & \alpha_{n+1} \\ & \vdots & \vdots & \vdots \\ \alpha_n & \alpha_{n+1} & \dots & \alpha_{2n-1} \end{bmatrix}. \tag{2.5.1}$$

Definition 2.5.2. Given a sequence of real numbers $S = \{x_1, x_2, ..., x_n\}$, the *Vandermonde matrix* is an $n \times n$ matrix with the following structure
$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}. \tag{2.5.2}$$

Definition 2.5.3. Given a sequence of real numbers $S = \{\beta_{-n}, ..., \beta_0, ..., \beta_n\}$, the *Toeplitz matrix* is an $(n+1) \times (n+1)$ matrix with the following structure

$$T = \begin{bmatrix} \beta_0 & \beta_1 & \dots & \beta_n \\ \beta_{-1} & \beta_0 & \dots & \beta_{n-1} \\ & \vdots & \vdots & \vdots \\ \beta_{-n} & \beta_{-(n-1)} & \dots & \beta_0 \end{bmatrix}. \tag{2.5.3}$$

Any non-singular Hankel matrix H can be factorized as two Vandermonde matrices and diagonal matrix $H = V^T DV$ i.e

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_2 & \alpha_3 & \dots & \alpha_{n+1} \\ & \vdots & \vdots & \vdots \\ \alpha_n & \alpha_{n+1} & \dots & \alpha_{2n-1} \end{bmatrix} =$$
$$\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_{n-1} & x_n \\ & \vdots & \vdots & \vdots & \\ x_1^{n-1} & x_2^{n-1} & \dots & x_{n-1}^{n-1} & x_n^{n-1} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} \tag{2.5.4}$$

Similarly, the Toeplitz matrix can be factorized as two Vandermonde matrices and diagonal matrix.

2.6 Generalized Shift Operators

1. Given the shift operator $\mathcal{S}_h : C(\mathbb{R}) \to C(\mathbb{R})$ with $\mathcal{S}_h f(x) = f(x-h)$, where $h > 0$, the symmetric shift operator is given by

$$\mathcal{S}_{h,-h} f(x) := \frac{1}{2}\big(f(x-h) + f(x+h)\big) = \frac{1}{2}\big(\mathcal{S}_{-h} + \mathcal{S}_h\big) f(x). \tag{2.6.1}$$

2. For a given continuous function $K : \mathbb{R}^2 \to \mathbb{C}$ such that

$$K(x, h_1 + h_2) = K(x, h_1)K(x + h_1, h_2) = K(x, h_2)K(x + h_2, h_1), \tag{2.6.2}$$

the shift operator

$$\mathcal{S}_{K,h} : C(\mathbb{R}) \to C(\mathbb{R})$$

where $h \neq 0$, can be defined by

$$\mathcal{S}_{K,h} f(x) = K(x, h) f(x + h). \tag{2.6.3}$$

2.7 Gaussian Integral

There are many nice definite integrals that can be used to solve a lot of problems. One of these integrals is the Gaussian integral, or probability integral. It is the improper integral of the Gaussian function, which is a widely used function in different fields, such as signal processing and statistics. The integral of the Gaussian function is very important for theory and practice, with the following property

$$\int_{-\infty}^{\infty} exp(-x^2)dx = \sqrt{\pi}. \tag{2.7.1}$$

The Gaussian integral can be generalized to have different forms, such as

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}dx = \sqrt{\frac{\pi}{a}e^{\frac{b^2}{4a}+c}}. \tag{2.7.2}$$

2.8 Chebyshev Polynomials

Chebyshev polynomials are very useful in different areas, such as numerical analysis and applied mathematics. We introduce the definition of Chebyshev polynomials of the first kind.

Definition 2.8.1. Chebyshev polynomials of degree $n \geq 0$ is defined as

$$\mathbf{T}_n(x) = \cos(n \arccos x), \quad x \in [-1, 1]. \tag{2.8.1}$$

There is a recursive relation of the Chebyshev polynomials with the following form

$$\mathbf{T}_{n+1}(x) = 2x\mathbf{T}_n(x) - \mathbf{T}_{n-1}(x). \tag{2.8.2}$$

Here are the first few Chebyshev polynomials of the first kind

$$\begin{aligned}
\mathbf{T}_0(x) &= 1 \\
\mathbf{T}_1(x) &= x \\
\mathbf{T}_2(x) &= 2x^2 - 1 \\
\mathbf{T}_3(x) &= 4x^3 - 3x \\
\mathbf{T}_4(x) &= 8x^4 - 8x^2 + 1 \\
\mathbf{T}_5(x) &= 16x^5 - 20x^3 + 5x \\
\mathbf{T}_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1.
\end{aligned} \tag{2.8.3}$$

2.9 Resultant and the Sylvester Matrix

The *resultant* is an important tool in algebra. It has been used as an alternative technique other than the *greatest common divisor* method to find the common zeros of two univariate polynomials. In this section, we introduce the definition of the resultant and the related Sylvester matrix.

Definition 2.9.1. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{2.9.1}$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \tag{2.9.2}$$

be two univariate polynomials of degree n and m with $n > 0$ and $m > 0$, respectively, such that their coefficients are in an arbitrary field $\mathbb{F}$. Then the resultant $\text{Res}(f, g)$ is defined as the smallest-degree polynomial of the variables $(a_j, j = 0, ..., n)$ and $(b_j, j = 0, ..., m)$ that vanishes if and only if $f(x)$ and $g(x)$ have a common zero.

Definition 2.9.2. The Sylvester matrix of the polynomials f and g defined in (2.9.1) and (2.9.2) is a square matrix with $n + m$ rows and columns. The first m rows contain shifted coefficient sequences of f and the last n rows contain shifted coefficient sequences of g, and it is given by

$$Syl(f, g) = \begin{bmatrix} a_n & a_{n-1} & \dots & a_0 & & & & \\ & a_n & a_{n-1} & \dots & a_0 & & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & a_n & a_{n-1} & \dots & a_0 & \\ b_m & b_{m-1} & & \dots & b_0 & & & \\ & b_m & b_{m-1} & & \dots & b_0 & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_m & b_{m-1} & & \dots & b_0 \end{bmatrix}. \tag{2.9.3}$$

The resultant $\text{Res}(f, g)$ is connected to the Sylvester matrix $Syl(f, g)$ as follows,

$$\text{Res}(f, g) = \det\left(Syl(f, g)\right). \tag{2.9.4}$$

Chapter 3

Prony Method and its Variations

The Prony method, introduced by Gaspard Riche de Prony in 1795 [**17**], is a commonly-used method to recover signals from given samples. Given $f(x) = \sum_{j=1}^{M} c_j e^{-ix\phi_j}$, in order to recover the frequencies ϕ_j and their corresponding coefficients c_j, Prony used equidistant sampling points $f(lh), l = 0, ..., 2M-1$. The Prony method has been applied in various fields, and its generalized version in terms of sparse expansions of eigenfunctions covers all the well-known Prony problems, which are viewed as its special cases [**12 15**]. Moreover, Prony Method can be generalized to reconstruct many types of expansions.

In sections (3.1) and (3.2) we describe the Prony method algorithm in details. We consider in case c_j are real, so we have $2M$ sampling values, while in the case when c_j are complex so we have fewer sampling value to recover the parameters of the expansion $f(x)$. In section (3.3), we describe generalized Prony method in terms of eigenfunctions of linear operators. Lastly, in sections (3.4) , (3.5) and (3.6), we describe some variations of this method.

3.1 Prony method with $2M$ sample values

The Prony method solves the following classic problem.

Problem P There is an unknown function $f(x)$ that is in the exponential form

$$f(x) = \sum_{j=1}^{M} c_j e^{-ix\phi_j} \tag{3.1.1}$$

with $M \geq 1$. A set of equispaced sampling values $f(lh), l = 0, ..., 2M-1$ are given, where h is some positive constant. How to recover the non-zero complex parameters c_j and distinct real-valued frequencies ϕ_j , $j = 1, ..., M$?

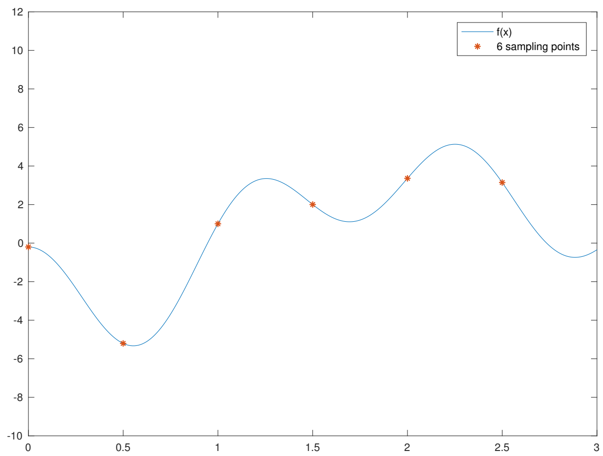

Figure 3.1: The blue line shows the function $f(x)$ in (3.2.9) while 6 red stars are the sampling values.

To solve this problem, we consider the Prony polynomial

$$\Lambda(z) = \prod_{j=1}^{M}(z - e^{-ih\phi_j}) = \sum_{l=0}^{M} \lambda_l z^l, \tag{3.1.2}$$

where $\lambda_j, j = 0, ..., M$ are the coefficients of the monomial terms in (3.1.2) with the leading coefficient $\lambda_M = 1$.

Let us consider the following linear homogeneous difference equation of order M for the expansion f in (3.1.1) for $m = 0, ..., M-1$, and we observe that

$$\begin{aligned} \sum_{l=0}^{M} \lambda_l f(h(l+m)) &= \sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j e^{-ih(l+m)\phi_j} = \sum_{j=1}^{M} c_j e^{-ihm\phi_j} \sum_{l=0}^{M} \lambda_l e^{-ihl\phi_j} \\ &= \sum_{j=1}^{M} c_j e^{-ihm\phi_j} \underbrace{\Lambda(e^{-ih\phi_j})}_{=0} = 0. \end{aligned} \tag{3.1.3}$$

Thus, we obtain the following linear system

$$\sum_{l=0}^{M-1} \lambda_l f(h(l+m)) = -f(h(M+m)), \quad m = 0, 1, ..., M-1. \tag{3.1.4}$$

From the given sampling values $f(lh), l = 0, ..., 2M-1$, the coefficient vectors $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{M-1})^T$ can be determined by solving the inhomogeneous system (3.1.4), or the following equivalent matrix form

$$\begin{bmatrix} f(0) & f(h) & \dots & f(h(M-1)) \\ f(h) & f(2h) & \dots & f(h(M)) \\ \vdots & \vdots & \vdots & \vdots \\ f(h(M-1)) & f(h(M)) & \dots & f(h(2M-2)) \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{M-1} \end{bmatrix} = - \begin{bmatrix} f(h(M)) \\ f(h(M+1)) \\ \vdots \\ f(h(2M-1)) \end{bmatrix} \tag{3.1.5}$$

Notice that the coefficient matrix of the system (3.1.5), denoted by $\boldsymbol{H} = (f(h(l+m))_{l,m=0}^{M-1}$, is an invertible matrix since it has a special Hankel structure, which allows us to factorize it as a matrix product of one diagonal matrix sandwiched by two Vandermonde-type matrices. Specifically,

$$\begin{aligned} \boldsymbol{H} &= \Big(f(h(l+m))\Big)_{l,m=0}^{M-1} \\ &= \Big(\sum_{j=0}^{M} c_j e^{-ih(l+m)\phi_j}\Big)_{l,m=0}^{M-1} \\ &= \Big(\sum_{j=0}^{M} c_j e^{-ihm\phi_j}.e^{-ihl\phi_j}\Big)_{l,m=0}^{M-1} \\ &= \Big(c_j e^{-ihm\phi_j}\Big)_{m=0,j=1}^{M-1}.\Big(e^{-ihl\phi_j}\Big)_{j=1,l=0}^{M-1} \\ &= \Big(e^{-ihm\phi_j}\Big)_{m=0,j=1}^{M-1} \text{daig}(c_1,...,c_M).\Big(e^{-ihl\phi_j}\Big)_{j=1,l=0}^{M-1} \\ &= \boldsymbol{V}^T \text{diag}(c_1,...,c_M)\boldsymbol{V}, \end{aligned} \tag{3.1.6}$$

where $\boldsymbol{V} = (e^{-ilh\phi_j})_{l,j=0}^{M-1}$ is a Vandermonde-type matrix, which is non-singular for distinct ϕ_j's, and $c_1, \dots c_M$ are non-zero.

By solving the system (3.1.5), we can find the coefficients λ_l of the Prony polynomial $\Lambda(z)$. It follows that the unknown frequencies can be extracted from the zeros of $\Lambda(z)$, represented as $z_j = e^{-ih\phi_j}$ with $h\phi_j \in (-\pi, \pi]$ for $j = 1, ..., M$. i.e

$$\phi_j = \frac{-\text{Im}(\ln(z_j))}{h}, \quad j = 1, ..., M. \tag{3.1.7}$$

Finally, the coefficients $c_j, j = 1, ..., M$ can be determined by solving the overdetermined Vandermonde linear system

$$f(lh) = \sum_{j=1}^{M} c_j e^{-ilh\phi_j}, \quad l = 0, ..., 2M-1 \tag{3.1.8}$$

or its equivalent matrix form

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{-ih\phi_1} & e^{-ih\phi_2} & \dots & e^{-ih\phi_M} \\ & \vdots & \vdots & \vdots \\ e^{-i(2M-1)h\phi_1} & e^{-i(2M-1)h\phi_2} & \dots & e^{-i(2M-1)h\phi_M} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} f(0) \\ f(h) \\ \vdots \\ f((2M-1)h) \end{bmatrix}. \tag{3.1.9}$$

Algorithm 1 Prony method with $2M$ sample values.

Input:

- Sampling values $f(lh), l = 0, ..., 2M - 1$.
- Choose sampling size h such that $h\phi_j \in (-\pi, \pi], \forall j \in \{1, ..., M\}$.

Calculation:

- Solve the inhomogeneous linear system

  $$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$

  where $\boldsymbol{H} = (f(h(l+m)))_{l,m=0}^{M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{M}$ and $\boldsymbol{F} = (f(h(l+M)))_{l=0}^{M-1}$.

- Find all the zeros $z_j := e^{-ih\phi_j}, j = 1, ..., M$ of the polynomial

  $$\Lambda(z) = \sum_{l=1}^{M} \lambda_l z^l,$$

  and then find $\phi_j, j = 1, \dots, M$ from z_j's.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the Vandermonde system

  $$\sum_{j=1}^{M} c_j e^{-ilh\phi_j} = f(lh), \quad l = 0, ..., 2M - 1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

3.2 Prony method with $M + 1$ sample values

In section(3.1), we showed that the coefficients c_j and the frequencies $\phi_j, j = 1, ..., M$ can be recovered by $2M$ equidistant sampling values where the coefficients $c_j, j = 1, ..., M$ are assumed to be complex parameters. If we know that the coefficients c_j's are all real, then we only need approximately one half of the sampling values, as described in the following problem.

Problem R There is an unknown function $f(x)$ that is in the exponential form

$$f(x) = \sum_{j=1}^{M} c_j e^{-ix\phi_j} \tag{3.2.1}$$

with $M \geq 1$. A set of equispaced sampling values $f(lh), l = 0, ..., M$ are given, where h is some positive constant. How to recover the non-zero *real* parameters c_j and distinct real-valued frequencies ϕ_j , $j = 1, ..., M$?

Observe that the expansion (3.2.1) satisfies the conjugate symmetry property

$$f(-x) = \sum_{j=1}^{M} c_j e^{ix\phi_j} = \overline{f(x)}, \tag{3.2.2}$$

and this property can help us to derive many sampling values from the given ones, more specifically, we can get $f(-lh)$ from $f(lh)$. Therefore, the coefficients c_j and the frequencies $\phi_j, j = 1, ..., M$ can be recovered by using fewer sampling values: $f(0), f(h), \ldots, f(Mh)$, which can still provide us the needed $2M + 1$ sampling values for the calculation.

Let us take our sampling values $f(lh), l = -M, ..., -1, 0, 1, ..., M$, by applying the conjugate symmetry property (3.2.2) the sampling values $f(lh), l = -M, ..., -1$ can be calculated from the sampling values $f(lh), l = 1, ..., M$ where h assumed to be a positive constant as before satisfies $h\phi_j \in (-\pi, \pi]$.

Thus, $M + 1$ sampling values are sufficient to recover all coefficients c_j and the frequencies $\phi_j, j = 1, ..., M$ in (3.2.1).

As in the section (3.1), we consider the Prony polynomial

$$\Lambda(z) = \prod_{j=1}^{M} (z - e^{-ih\phi_j}) = \sum_{l=0}^{M} \lambda_l z^l, \tag{3.2.3}$$

where $\lambda_j, j = 1, ..., M$ are the coefficients of the monomial representation (3.2.3) with $\lambda_M = 1$.

Let us consider again the linear homogeneous difference equation of order M for f in (3.2.1), then for all $m = 0, ..., M-1$ and $\lambda_M = 1$, we observe that

$$\begin{aligned}\sum_{l=0}^{M} \lambda_l f(h(l-m)) &= \sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j e^{-ih(l-m)\phi_j} = \sum_{j=1}^{M} c_j e^{ihm\phi_j} \sum_{l=0}^{M} \lambda_l e^{-ihl\phi_j} \\ &= \sum_{j=1}^{M} c_j e^{ihm\phi_j} \underbrace{\Lambda(e^{-ih\phi_j})}_{=0} = 0.\end{aligned} \tag{3.2.4}$$

Thus, we obtain the following linear system

$$\sum_{l=0}^{M-1} \lambda_l f(h(l-m)) = -f(h(M-m)), \quad m = 0, 1, ..., M-1. \tag{3.2.5}$$

or its equivalent matrix form

$$\begin{bmatrix} \overline{f(0)} & f(h) & \dots & f((M-1)h) \\ f(h) & f(0) & \dots & f((M-2)h) \\ & \vdots & \vdots & \vdots \\ \overline{f((M-1)h)} & \overline{f((M-2)h)} & \dots & f(0) \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{M-1} \end{bmatrix} = - \begin{bmatrix} f(h(M)) \\ f(h(M-1)) \\ \vdots \\ f(1) \end{bmatrix}. \tag{3.2.6}$$

We also note that the coefficient matrix of the system (3.2.6), $\boldsymbol{T} = (f(h(l-m)))_{l,m=0}^{M-1}$, is no longer a Hankel matrix as in the section (3.1), but it has a Toeplitz structure, which is also invertible. The matrix $\boldsymbol{T}$ can be factorized into a matrix product of a diagonal matrix sandwiched by two Vandermonde-type matrices as follow:

$$\begin{aligned}\boldsymbol{T} &= \Big(f(h(l-m)) \Big)_{l,m=0}^{M-1} \\ &= \left(\sum_{j=0}^{M} c_j e^{-ih(l-m)\phi_j} \right)_{l,m=0}^{M-1} \\ &= \left(\sum_{j=0}^{M} c_j e^{ihm\phi_j} . e^{-ihl\phi_j} \right)_{l,m=0}^{M-1} \\ &= \Big(c_j e^{ihm\phi_j} \Big)_{m=0,j=1}^{M-1} . \Big(e^{-ihl\phi_j} \Big)_{j=1,l=0}^{M-1}\end{aligned}$$

$$\begin{aligned} &= \left(e^{ihm\phi_j}\right)_{m=0,j=1}^{M-1} \text{daig}(c_1, ..., c_M). \left(e^{-ihl\phi_j}\right)_{j=1,l=0}^{M-1} \\ &= \boldsymbol{V}^* \text{diag}(c_1, ..., c_M) \boldsymbol{V}, \end{aligned} \tag{3.2.7}$$

where $\boldsymbol{V} = (e^{-ilh\phi_j})_{l,j=0}^{M-1}$ is a Vandermonde-type matrix.

Again the solution of the system (3.2.6) provides us the coefficients λ_l of the Prony polynomial $\Lambda(z)$, and by using (3.1.7) we can extract the frequencies ϕ_j.

Finally, the coefficients $c_j, j = 1, ..., M$ can be determined by solving the linear system

$$f(lh) = \sum_{j=1}^{M} c_j e^{-ilh\phi_j}, \quad l = 0, ..., M. \tag{3.2.8}$$

Algorithm 2 Prony method with $M + 1$ sample values.

Input:

- Sampling values $f(lh), l = 0, ..., M$.
- Choose sampling size h such that $h\phi_j \in (-\pi, \pi], \forall j \in \{1, ..., M\}$.

Calculation:

- Calculate $f(-lh), l = 1, ..., M$ using $f(-lh) = \overline{f(lh)}$.
- Solve the inhomogeneous linear system

$$\boldsymbol{T}\boldsymbol{\lambda} = -\boldsymbol{F},$$

  where $\boldsymbol{T} = (f(h(l-m)))_{l,m=0}^{M-1}$ is a Toeplitz matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{M}$ and $\boldsymbol{F} = (f(h(M-l)))_{l=0}^{M-1}$.
- Find all the zeros $z_j := e^{ih\phi_j}, j = 1, ..., M$ of the polynomial

$$\Lambda(z) = \sum_{l=1}^{M} \lambda_l z^l,$$

  and calculate $\phi_j, j = 1, \ldots, M$ from z_j's.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the Vandermonde system

$$\sum_{j=1}^{M} c_j e^{-ilh\phi_j} = f(lh), \quad l = 0, ..., M.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

To illustrate the algorithm, let us consider a simple signal

$$f(x) = f_1(x) + f_2(x) + f_3(x) \tag{3.2.9}$$

with three components

$$\begin{aligned} f_1(x) &= c_1 e^{-ix\phi_1}, \quad c_1 = -3.5, \quad \phi_1 = 1.5; \\ f_2(x) &= c_1 e^{-ix\phi_2}, \quad c_2 = 0.8, \quad \phi_2 = 4; \\ f_3(x) &= c_1 e^{-ix\phi_3}, \quad c_3 = 2.5, \quad \phi_3 = 5.5. \end{aligned} \tag{3.2.10}$$

The table (3.1) shows the absolute reconstruction errors $|c_j - c_j^*|$ and $|\phi_j - \phi_j^*|$ where c_j^* and ϕ_j^* are the reconstructed parameters and frequencies respectively.

j	c_j	ϕ_j	$\lvert c_j - c_j^*\rvert$	$\lvert\phi_j - \phi_j^*\rvert$
1	-3.5	1.5	$5.6344.10^{-15}$	$6.6613.10^{-16}$
2	0.8	4	$1.2610.10^{-15}$	$7.1054.10^{-15}$
3	2.5	5.5	$8.0351.10^{-15}$	$3.5527.10^{-15}$

Table 3.1: Parameters of the function $f(x)$ in (3.2.9) and approximate errors with $h = 0.5$.

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3.3 Prony method for sparse expansion of eigenfunctions

In this section, we review the generalized Prony method [**12**]. The Prony method can be generalized to reconstruct different expansion functions.

Given a normed vector space $\mathcal{V}$ over the complex number field $\mathbb{C}$, and given a linear operator

$$\mathcal{A} : \mathcal{V} \to \mathcal{V}, \tag{3.3.1}$$

where $\mathcal{A}$ is assumed to possess eigenvalues. Assume that $\Lambda = \{\lambda_j, j \in I\}$ is a (sub)set of pairwise distinct eigenvalues of $\mathcal{A}$ with suitable index set I. We also assume $\mathcal{W} = \{v_j : j \in I\}$ such that there is a unique correspondence between eigenvalues λ_j and eigenfunction v_j, i.e

$$\mathcal{A}v_j = \lambda_j v_j, \quad \text{for all} \quad j \in I. \tag{3.3.2}$$

The M-sparse expansion f of eigenfunctions of the operator $\mathcal{A}$ has only M-non vanishing terms, i.e

$$f = \sum_{j \in J} c_j v_j, \quad \text{with} \quad J \subset I, \text{and} \quad |J| = M. \tag{3.3.3}$$

Moreover, we assume that there exists a linear functional

$$\mathcal{F} : \mathcal{V} \to \mathbb{C} \tag{3.3.4}$$

such that $\mathcal{F}v_j \neq 0, \forall j \in I$. Then, by the Prony method (see [**12**]), the expansion f in (3.3.3) can be uniquely reconstructed from $2M$ sampling values, $\mathcal{F}(\mathcal{A}^l f), l = 0, ..., 2M-1$.

Since the above setting is very general, we can use it as a framework to solve a large number of sparse expansion problems. We would like to formulate it as the following problem.

Problem O There is an unknown function $f(x)$ that is in an expansion of the eigenfunctions $\{v_j, j \in J\}$ of a linear operator $\mathcal{A}$ as follows,

$$f(x) = \sum_{j \in J}^{M} c_j v_j. \tag{3.3.5}$$

Assume that there exists a linear functional $\mathcal{F} : \mathcal{V} \to \mathbb{C}$ with $\mathcal{F}v_j \neq 0, \forall j \in I$. How to recover the non-zero complex parameters c_j from $2M$ sampling values, $\mathcal{F}(\mathcal{A}^l f), l = 0, ..., 2M-1$?

As in the classical Prony method we introduce Prony polynomial

$$P(z) = \prod_{j \in J}(z - \lambda_j) = \sum_{l=0}^{M} p_l z^l, \tag{3.3.6}$$

where the zeros $\lambda_j, j \in J$ of the $P(z)$ in (3.3.6) are the eigenvalues corresponding to the active eigenfunctions $v_j, j \in J$ with $p_M = 1$. By using the expansion f in (3.3.5), and for $m = 0, 1..., M-1$, we can see that

$$\begin{aligned}\sum_{l=0}^{M} p_l \mathcal{F}(\mathcal{A}^{l+m} f) &= \sum_{l=0}^{M} p_l \mathcal{F}\left(\sum_{j\in J} c_j \lambda_j^{l+m} v_j\right) = \sum_{j\in J} c_j \lambda_j^{m} \left(\sum_{l=0}^{M} p_l \lambda_j^{l}\right) \mathcal{F} v_j \\ &= \sum_{j\in J} c_j \lambda_j^{m} \underbrace{P(\lambda_j)}_{=0} \mathcal{F} v_j = 0.\end{aligned} \tag{3.3.7}$$

Thus, we obtain the following linear system

$$\sum_{l=0}^{M-1} p_l \mathcal{F}(\mathcal{A}^{l+m} f) = -\mathcal{F}(\mathcal{A}^{M+m} f), \quad m = 0, ..., M-1. \tag{3.3.8}$$

Therefore, the vector of coefficients $\boldsymbol{p} = (p_0, ..., p_{M-1})^T$ can be determined by solving the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{p} = -\boldsymbol{G}, \tag{3.3.9}$$

where $\boldsymbol{G} := (\mathcal{F}(\mathcal{A}^{M+m} f))_{m=0}^{M-1}$, and $\boldsymbol{H} := (\mathcal{F}(\mathcal{A}^{l+m} f))_{l,m=0}^{M-1,M-1}$, and the matrix $\boldsymbol{H}$ has a Hankel structure, which is invertible since it can be written as

$$\boldsymbol{H} = \boldsymbol{V}_{\lambda} \mathrm{diag}(c_j)_{j\in J}.\mathrm{diag}(\mathcal{F} v_j)_{j\in J} \boldsymbol{V}_{\lambda}^T, \tag{3.3.10}$$

where $\boldsymbol{V} = (\lambda^l)_{l=0,j\in J}^{M-1}$ is of the Vandermonde-type.

By finding the coefficients $(\boldsymbol{p}_l, l = 0, ..., M-1)$, we can extract the eigenvalues $\lambda_j, j = 0, ..., M-1$ from the Prony polynomial (3.3.6), and since $\mathcal{A} v_j = \lambda_j v_j$, the eigenfunctions v_j can be determined.

Finally, the coefficients $c_j, j \in J$ of the expansion (3.3.3) can be computed by solving the Vandermonde linear system

$$\mathcal{F}(\mathcal{A}^l f) = \sum_{j\in J} c_j \lambda_j^l v_j, \quad l = 0, ..., 2M-1. \tag{3.3.11}$$

Algorithm 3 Prony method for the sparse eigenfunction expansion (3.3.5).

Input:

- $M \in \mathbb{N}$.
- Sampling values $\mathcal{F}(\mathcal{A}^l f), l = 0, ..., 2M-1$.

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{p} = -\boldsymbol{G}, \tag{3.3.12}$$

  where $\boldsymbol{H} := (\mathcal{F}(\mathcal{A}^{l+m}f))_{l,m=0}^{M-1,M-1}$ and $\boldsymbol{G} := (\mathcal{F}(\mathcal{A}^{M+m}f))_{m=0}^{M-1}$.

- Find all the zeros $\lambda_j, j = 1, ..., M$ of the Prony polynomial (3.3.6), and then compute the corresponding eigenfunctions $v_j, j \in J$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the Vandermonde system

$$\mathcal{F}(\mathcal{A}^l f) = \sum_{j \in J} c_j \lambda_j^l v_j, \quad l = 0, ..., 2M-1.$$

Output:

- c_j and $v_j, j \in J$.

3.4 Non Uniform Spline Expansion

The original Prony method used to determine the parameters and the frequencies of the exponential sum. In [**19**, **30**] a nice technique has been used to convert the B-spline expansion to the exponential sum and then apply the original Prony method to determine the coefficients and the frequencies. In this section, we review this technique.

Let

$$f(x) = \sum_{j=1}^{M} c_j^m N_j^m(x)$$

where $N_j^m(x)$ are the B-spline of order m as defined in (2.4.1) of section(2.4). We consider the following spline expansion problem.

Problem B_1 There is an unknown function $f(x)$ that is in the spline expansion form

$$f(x) = \sum_{j=1}^{M} c_j N_j^m(x) \tag{3.4.1}$$

with $M \geq 1$. Assume that the B-splines $\{N_j^m(x), j = 1, 2, \ldots, M\}$ are defined on the knot sequence $\{\phi_1, \phi_2, \ldots, \phi_{M+m}\}$ on the interval $[a, b]$ satisfying

$$a = \phi_1 = \cdots = \phi_m < \phi_{m+1} < \cdots < \phi_M < \phi_{M+1} = \cdots = \phi_{M+m} = b.$$

If we are given the $M + m$ Fourier sampling values $\hat{f}(lh), l = 1, ..., M + m$, where h is a positive constant satisfying $h\phi_j \in (-\pi, \pi]$, how to recover the non-zero real parameters $c_j, j = 1, 2, \ldots, M$ and distinct knots $\phi_{m+1}, \ldots, \phi_M$?

3.4.1 Reconstruction of Characteristic Expansion

Let us consider the expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j^1 N_j^1(x), \tag{3.4.2}$$

where $N_j^1(x)$ are characteristic functions as defined in (2.4.2), and $c_j^1, j = 1, .., N$ are distinct real coefficients.

By applying Fourier transform to (3.4.2), discussed in section(2.2), we can get (see [**19**, **30**]).

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{i\omega}\left[\sum_{j=1}^{M} c_j^1 e^{-i\omega T_j} - \sum_{j=2}^{M+1} c_{j-1}^1 e^{-i\omega T_j}\right] \\ &= \frac{1}{i\omega}\sum_{j=1}^{M+1} c_j^{[1]} e^{-i\omega T_j}, \quad \omega \neq 0. \end{aligned} \tag{3.4.3}$$

with $c_j^{[1]} := c_j^1 - c_{j-1}^1, j = 1, ..., M + 1$, and with $c_0^1 = c_{M+1}^1 = 0$, from the assumption that $c_j^{[1]} \neq 0, j = 1, ..., M + 1$.

Let

$$P(\omega) = (i\omega)\hat{f}(\omega) = \sum_{j=1}^{M+1} c_j^{[1]} e^{-i\omega\phi_j} \tag{3.4.4}$$

Prony method can be applied to the expansion (3.4.4) to recover the coefficients c_j^1 and the frequencies $\phi_j, j = 1, ..., M + 1$ uniquely. Since the coefficients $c_j, j = 1, ..., M + 1$ assumed to be real as in the section(3.2), then we have

$$\begin{aligned} P(lh) &= (ilh) \cdot \hat{f}(lh), \quad l = 1, ..., M + 1, \\ P(-lh) &= \overline{P(lh)}, \quad l = 1, ..., M + 1, \\ P(0) &= 0, \end{aligned} \tag{3.4.5}$$

where h is assumed to be a positive constant that satisfies $h\phi_j \in (-\pi, \pi]$.

Finally, the coefficients $c_j^1, j = 0, ..., M$ can be obtained by using the following recursion

$$\begin{aligned} c_1^1 &= c_1^{[1]}, \\ c_j^1 &= c_{j-1}^1 + c_j^{[1]}, \quad j = 2, ..., M. \end{aligned} \tag{3.4.6}$$

3.4.2 Reconstruction of Non Uniform Spline Expansion

Let us now consider the spline expansion when $m \geq 1$

$$f(x) = \sum_{j=1}^{M} c_j^{[0]} N_j^m(x) \tag{3.4.7}$$

In order to apply the Fourier transform and recover the parameters $c_j^{[0]}$ and knots ϕ_j of the expansion (3.4.7), we need to compute the derivatives of B-spline functions.

The first derivatives of B-spline functions N_j^m when $m \geq 3$ are given by (see [**30**] page 23).

$$(N_j^m)'(x) = (m-1) \cdot \left(\frac{N_j^{m-1}(x)}{\phi_{j+m-1} - \phi_j} - \frac{N_{j+1}^{m-1}(x)}{\phi_{j+m} - \phi_{j+1}} \right). \tag{3.4.8}$$

Indeed the kth derivative of the expansion (3.4.7) can be computed by

$$f^{(k)}(x) = \sum_{j=1}^{M+k} c_j^{[k]} N_j^{m-k}(x), \tag{3.4.9}$$

where the coefficients $c_j^{[k]}, j = 1, ...M + k$ can be recursively computed by

$$c_j^{[k]} := \left(\frac{m-k}{\phi_{j+m-k} - \phi_j} \right) \cdot (c_j^{[k-1]} - c_{j-1}^{[k-1]}) \tag{3.4.10}$$

with $c_0^{[k-1]} = c_{M+k}^{[k-1]} = 0$.

Note that when $k = m-2$, the kth derivative $f^{(k)}$ of spline function is piecewise linear functions which is not differentiable at the knots. In order to compute the derivatives of B-spline of order one and two,i.e ($m = 1, m = 2$), we need to take the distributional derivative (see [**30**] lemma 3.5 and lemma 3.6, page 25) .

The m-th derivative of the expansion f in (3.4.7) can be derived as a linear combination of weighted Dirac distributions

$$f^{(m)}(x) = \sum_{j=1}^{M+m} c_j^{[m]} \delta(x - \phi_j). \tag{3.4.11}$$

Hence, by using the properties of Fourier transform discussed in section (2.2), then the Fourier transform of (3.4.11) can be given by

$$P(\omega) = (i\omega)^m \hat{f}(\omega) = \sum_{j=1}^{M+m} c_j^{[m]} e^{-i\omega\phi_j} \tag{3.4.12}$$

The coefficients $c_j^{[m]}$ and the frequencies $\phi_j, j = 1, ..., M + m$ of the expansion sum (3.4.12) can be uniquely recovered by using $M + m$ Fourier sampling values $\hat{f}(lh), l = 1, ..., M + m$, where h is assumed to be a positive constant satisfies $h\phi_j \in (-\pi, \pi]$.

Indeed, as in the section(3.2), $P(\omega)$ has the conjugate symmetry property since $c_j^{[0]}, j = 1, ..., M + m$ are assumed to be real-valued coefficients. Then we have

$$P(-\omega) = (-i\omega)^m \hat{f}(-\omega) = \overline{((i\omega)^m \hat{f}(\omega)} = \overline{P(\omega)}. \tag{3.4.13}$$

Therefore, we need just $M + m$ Fourier sampling values in order to recover all the parameters. Also, one can verify that $P(0) = 0$.

Finally, the coefficients $c_j^{[0]}, j = 1, ..., M$ can be computed by the following recursion formula

$$c_j^{[k-1]} = \begin{cases} c_1^{[m]} & \text{for} \quad k = m, j = 1, \\ c_j^{[m]} + c_{j-1}^{[m-1]} & \text{for} \quad k = m, j = 2, ..., M + m - 1, \\ \left(\frac{\phi_{1+m-k} - \phi_1}{m-k}\right) c_1^{[k]} & \text{for} \quad k = m - 1, ..., 1, j = 1, \\ \left(\frac{\phi_{j+m-k} - \phi_j}{m-k}\right) c_j^{[k]} + c_{j-1}^{[k+1]} & \text{for} \quad k = m - 1, ..., 1, j = 2, ..., M + k - 1. \end{cases} \tag{3.4.14}$$

3.5 Reconstruction of Cosine Expansions Using the Chebyshev Polynomial

Let us consider the expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x), \tag{3.5.1}$$

where c_j's are non-zero coefficients and ϕ_j's are real-valued frequencies for $j = 1, ..., M$. We want to reconstruct c_j and ϕ_j in (3.5.1), with the frequencies ϕ_j assumed to be in the range $[0, K) \subset \mathbb{R}$ and $h = \frac{\pi}{K}$, [**17**].

Using the symmetric shift operator defined in (2.6.1), we have

$$\mathcal{S}_{h,-h} \cos(\phi x) = \frac{1}{2}\Big[\cos(\phi(x-h)) + \cos(\phi(x+h))\Big] = \cos(\phi h)\cos(\phi x). \quad (3.5.2)$$

We observe that $\{\cos(\phi h), \phi \in \mathbb{C}\}$ is a set of distinct eigenvalues of the symmetric shift operator $\mathcal{S}_{h,-h}$.

The expansion f in (3.5.1) can be uniquely reconstructed using $2M$ sampling values $f(kh + x_0)$ for $k = 0, ..., 2M - 1$, and $x_0 \in \mathbb{R}$.

The Prony Polynomial can be defined as

$$\Lambda(z) = \prod_{j=1}^{M}(z - \cos(h\phi_j)). \quad (3.5.3)$$

This polynomial can be written in terms of the Chebyshev polynomial of the first kind of degree l as follows

$$\Lambda(z) = \sum_{l=0}^{M} \lambda_l \mathbf{T}_l(z), \quad (3.5.4)$$

where $\mathbf{T}_l(z) := \cos(l \cos^{-1}(z))$. We note that the leading coefficient of the Chebyshev polynomial $\mathbf{T}_l(z)$ is 2^{l-1}, then by the definition of the Prony polynomial Λ, we have $\lambda_M = 2^{1-M}$.

Then we observe for $m = 0, ..., M - 1$ that

$$\begin{aligned}\sum_{l=0}^{M} \lambda_l \Big((\mathcal{S}_{lh,-lh})\mathcal{S}_{mh} f(x_0) \Big) &= \frac{1}{2}\sum_{l=0}^{M} \lambda_l \Big(f(x_0 + (m+l)h + f(x_0 + (m-l)h) \Big) \\ &= \frac{1}{2}\sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j \Big(\cos(\phi_j(x_0 + (m+l)h)) + \cos(\phi_j(x_0 + (m-l)h)) \Big) \\ &= \sum_{j=1}^{M} c_j \cos(\phi_j(x_0 + mh)) \sum_{l=0}^{M} \lambda_l \cos(\phi_j l h)\end{aligned}$$

$$= \sum_{j=1}^{M} c_j \cos(\phi_j(x_0 + mh)) \underbrace{\sum_{l=0}^{M} \lambda_l \mathbf{T}_l(\cos(\phi_j lh))}_{=0} = 0. \tag{3.5.5}$$

Then for $x_0 = 0$ we have the following linear system

$$\sum_{l=0}^{M} \lambda_l \Big(f(x_0+(m+l)h+f(x_0+(m-l)h) \Big) = -2^{1-M} f(x_0+(M+l)h+f(x_0+(M-l)h). \tag{3.5.6}$$

We also note that the signal f is an even function, therefore the sampling values $f(kh), k = 0, ..., 2M-1$ are enough to build this system.

$$\begin{aligned} \mathbf{H} &= \Big(f((m+l)h) + f((m-l)h) \Big)_{m,l=0}^{M-1} \\ &= 2\Big(\sum_{j=1}^{M} c_j \cos(\phi_j mh) \cos(\phi_j lh) \Big)_{m,l=0}^{M-1} \\ &= 2\mathbf{V}\mathrm{diag}(c_j)_{j=1}^{M} \mathbf{V}^T. \end{aligned} \tag{3.5.7}$$

We observe that the terms $\cos(\phi_j h)$ are non-zero and distinct, therefore the matrix $\mathbf{H}$ is always invertible.

For $x_0 \neq 0$, we have also invertible matrix, (see [**17**]), but we need $4M-1$ sampling values $f(x_0 + hk)$, $k = -2M+1, ..., 2M-1$ for $x_0 \in \mathbb{R}$ and $\phi_j h \neq (2k+1)\frac{\pi}{2}$ to recover the parameters and the frequencies in (3.5.1).

By having the coefficients of the Prony polynomial, we can extract the zeros $\cos(h\phi_j), j = 1, ..., M$. Finally, the coefficients $c_j, j = 1, ..., M$ can be found by solving the following linear system

$$f(x_0 + hk) = \sum_{j=1}^{M} c_j \cos(\phi_j(x_0 + hk)), \quad k = 0, ..., 2M-1. \tag{3.5.8}$$

3.6 Reconstruction of Expansion of shift Gaussian

We end this chapter by reviewing the reconstruction of the shifted Gaussian, which has the form [**16**]

$$f(x) = \sum_{j=1}^{M} c_j g(x - \phi_j) = \sum_{j=1}^{M} c_j e^{-\beta(x-\phi_j)^2}, \tag{3.6.1}$$

where $\beta \in \mathbb{C}\backslash\{0\}$, has been reconstructed by using the generalized shift operator defined in (2.6.3).

Let us consider $K(x,h) := e^{\beta h(2x+h)}$, where $\beta \in \mathbb{C}\backslash\{0\}$, such that K satisfied (2.6.2), then by (2.6.3) we can see

$$(\mathcal{S}_{K,h} e^{-\beta(\phi_j - .)^2})(x) = e^{\beta h(2x+h)} e^{-\beta(\phi_j-(x+h))^2} = e^{2\beta\phi_j h} e^{-\beta(\phi_j - x)^2}. \tag{3.6.2}$$

Therefore $e^{-\beta(.-\phi_j)}$'s are eigenfunctions of $\mathcal{S}_{K,h}$ for all $\phi_j \in \mathbb{R}$.

The expansion f in (3.6.1) can be reconstructed using $2M$ sampling values $f(x_0 + hk), k = 0, ..., 2M-1$, $x_0 \in \mathbb{R}$ is an arbitrary real number. If $\mathrm{Re}\beta \neq 0$, then $h \in \mathbb{R}\backslash\{0\}$, while if $\mathrm{Re}\beta = 0$ then $0 < h \leq \frac{\pi}{2|\mathrm{Im}\beta|L}$, where $\phi_j \in (-L, L)$ for $j = 1, ..., M$ for some given L.

The Prony polynomial can be defined as:

$$\Lambda(z) = \prod_{j=1}^{M} (z - e^{2h\beta\phi_j}) = \sum_{j=0}^{M} \lambda_l z^l, \tag{3.6.3}$$

where λ_l are the coefficients of the monomial representation $\Lambda(z)$ with $\lambda_M = 1$. Then we have the following linear system

$$\begin{aligned} &\sum_{l=0}^{M-1} \lambda_l e^{\beta h(l+m)(2x_0+h(l+m))} f(x_0 + h(l+m)) \\ &\qquad = -e^{\beta h(l+M)(2x_0+h(l+M))} f(x_0 + h(l+M)) \quad m = 0, 1, ..., M-1. \end{aligned} \tag{3.6.4}$$

Therefore, the vector of the coefficients $\boldsymbol{\lambda} := (\lambda_0, ..., \lambda_{M-1})^T$ can be obtained by solving the inhomogeneous system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{G}, \tag{3.6.5}$$

where $\boldsymbol{G} := \left((\mathcal{S}_{K,(M+m)h} f)(x_0)\right)_{m=0}^{M-1}$, and $\boldsymbol{H} := \left((\mathcal{S}_{K,(l+m)h} f)(x_0+(l+m)h)\right)_{l,m=0}^{M-1,M-1}$, which is a Hankel matrix and invertible.

$$
\begin{aligned}
\boldsymbol{H} :&= \big((\mathcal{S}_{K,(l+m)h}f)(x_0+(l+m)h)\big)_{l,m=0}^{M-1} \\
&= \Big(K(x_0,(l+m)h)f(x_0+(l+m)h\Big)_{l,m=0}^{M-1} \\
&= \Big(e^{\beta h(l+m)(2x_0+(l+m)h)}\sum_{j=1}^{M} c_j e^{-\beta(x_0+(l+m)h-\phi_j)^2}\Big)_{l,m=0}^{M-1} \\
&= \Big(\sum_{j=1}^{M} c_j e^{-\beta(x_0-\phi_j)^2} e^{2\beta(l+m)h\phi_j}\Big)_{l,m=0}^{M-1} \\
&= \boldsymbol{V}\,\mathrm{diag}(c_j e^{-\beta(\phi_j-x_0)^2})\boldsymbol{V}^T,
\end{aligned}
\tag{3.6.6}
$$

with the Vandermonde matrix

$$
\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{2\beta h\phi_1} & e^{2\beta h\phi_2} & \dots & e^{2\beta h\phi_M} \\ \vdots & \vdots & \dots & \vdots \\ e^{2(M-1)\beta h\phi_1} & e^{2(M-1)\beta h\phi_2} & \dots & e^{2(M-1)\beta h\phi_M} \end{bmatrix}.
$$

Finally, the coefficients c_j of the expansion (3.6.1) can be computed by solving the following linear system:

$$
f(x_0+lh) = \sum_{j=1}^{M} c_j e^{-\beta(x_0-\phi_j+lh)^2}, \quad l = 0, ..., 2M-1. \tag{3.6.7}
$$

Chapter 4

Generalization of Prony Method

In the recent years, Prony method has been generalized for reconstructing different signal models using different techniques. By using suitable linear operator, signal models can be represented as sparse expansions of eigenfunctions. In this chapter, we provide some new generalizations of the Prony method.

In sections (4.1), (4.3) and (4.4) we use frequency domain in order to reconstruct the exponential sums (3.6.1), the cosine expansions (4.3.1) and non-stationary signal (4.4.1). In section (4.5) we use variable substitution to recover some expansions such as Gaussian expansion. In section (4.6), we use the polynomial of differential operator to reconstruct the exponential sums and the cosine expansions. In section (4.7), we reconstruct the Gaussian expansion using differential operator. In section (4.8), we define a new operator in order to reconstruct a new model (4.8.1). Lastly, in section (4.9),we provide some numerical experiments.

4.1 Reconstruction of Exponential Sums Using Short Time Fourier Transform

Let us consider a function $f(x)$ to be recovered in the exponential form

$$f(x) = \sum_{j=1}^{M} c_j e^{ix\phi_j} \tag{4.1.1}$$

for non-zero complex parameters c_j and distinct real-valued frequencies ϕ_j, $j = 1, ..., M$ with $M \geq 1$.

We use the frequency domain to recover the parameters c_j and frequencies ϕ_j , $j = 1, ..., M$. First we compute the STFT of (4.1.1) as follows:

$$\begin{aligned} STFT(\omega,\tau) &= \int_{-\infty}^{\infty}\left(\sum_{j=1}^{M} c_j e^{ix\phi_j}\right) w(x-\tau)e^{-i\omega x}dx \\ &= \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{ix\phi_j} w(x-\tau)e^{-i\omega x}dx \end{aligned} \tag{4.1.2}$$

By using a Gaussian window defined by

$$w(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{4.1.3}$$

then we observe that

$$\begin{aligned} STFT(\omega,\tau) &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2\sigma^2}+i(\phi_j-\omega)x}dx \\ &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}+i(\phi_j-\omega-\frac{i\tau}{\sigma^2})x-\frac{\tau^2}{2\sigma^2}}dx \\ &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{M} c_j \sqrt{2\pi\sigma^2} e^{-\frac{\sigma^2}{2}(\phi_j-\omega-i\tau/\sigma^2)^2-\frac{\tau^2}{2\sigma^2}} \\ &= \sigma \sum_{j=1}^{M} c_j e^{-\frac{\sigma^2}{2}(\phi_j-\omega-i\tau/\sigma^2)^2-\frac{\tau^2}{2\sigma^2}} \end{aligned} \tag{4.1.4}$$

Assume that $\tau = 0$ and $\sigma = 1$ we have

$$P(\omega) = STFT(\omega, 0) = \sum_{j=1}^{M} c_j e^{-\frac{1}{2}(\phi_j-\omega)^2}, \tag{4.1.5}$$

This model is related to the model in (3.6.1) when $\beta = \frac{1}{2}$. Therefore, the coefficients c_j and the frequencies ϕ_j in the expansion (4.1.1) can be recovered by using $2M$ short time Fourier sampling $P(\omega), \omega = 0, ..., 2M-1$.

Algorithm 4 Reconstruction of exponential sums using short time Fourier transform (4.1.1).

Input:

- Number of terms M.
- Short time Fourier sampling, $P(\omega_0 + lh), l = 0, ..., 2M-1$,using (4.1.5).
- Choose sampling size h.

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$

where $\boldsymbol{H} := \big((\mathcal{S}_{K,(l+m)h}P)(\omega_0)\big)_{l,m=0}^{M-1,M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{M-1}$ and $\boldsymbol{F} := \big((\mathcal{S}_{K,(M+m)h}P)(\omega_0)\big)_{m=0}^{M-1}$.

- Find all the zeros $z_j := e^{h\phi_j}, j = 1, ..., M$ of the polynomial

$$\Lambda(z) = \sum_{l=1}^{M} \lambda_l z^l,$$

and calculate ϕ_j from z_j for all $j = 1, \ldots, M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

$$P(\omega_0 + lh) = \sum_{j=1}^{M} c_j e^{-\frac{1}{2}(\phi_j+\omega_0-lh)^2}, \quad l = 0, ..., 2M-1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

4.2 Reconstruction of Mixed Expansions Using Generalized Shifted Operator

In order to reconstruct cosine expansions that will be introduced in the next two sections, we first reconstruct the mixed expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j e^{-\beta(\phi_j-x)^2} + \sum_{j=1}^{M} c_j e^{-\beta(\phi_j+x)^2} \tag{4.2.1}$$

to recover the unknown coefficients $c_j \in \mathbb{C}$ and $\phi_j \in \mathbb{R}$, $j = 1, ..., M$.

Let us consider $K(x, h) := e^{\beta h(2x+h)}$, where $\beta \in \mathbb{C}\backslash\{0\}$, such that K satisfies (2.6.2). Then by using the operator $\mathcal{S}_{K,h}$ defined in (2.6.3) we have the following properties:

$$(\mathcal{S}_{K,h}e^{-\beta(\phi_j-\cdot)^2})(x) = e^{\beta h(2x+h)}e^{-\beta(\phi_j-(x+h))^2} = e^{2\beta\phi_j h}e^{-\beta(\phi_j-x)^2}, \tag{4.2.2}$$

$$(\mathcal{S}_{K,h}e^{-\beta(\phi_j+\cdot)^2})(x) = e^{\beta h(2x+h)}e^{-\beta(\phi_j+(x+h))^2} = e^{2\beta\phi_j h}e^{-\beta(\phi_j+x)^2}. \tag{4.2.3}$$

Clearly $e^{-\beta(\cdot-\phi_j)}$ and $e^{-\beta(\cdot+\phi_j)}$ are eigenfunctions of $\mathcal{S}_{K,h}$ for all $\phi_j \in \mathbb{R}$.

The expansion f in (4.2.1) can be reconstructed using $4M$ sampling values $f(x_0+hk), k = 0, ..., 4M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number. If $\mathrm{Re}\beta \neq 0$, then $h \in \mathbb{R}\backslash\{0\}$, while if $\mathrm{Re}\beta = 0$ then $0 < h \leq \frac{\pi}{2|\mathrm{Im}\beta|L}$, where $\phi_j \in (-L, L)$ for $j = 1, ..., M$ for some given L.

The Prony polynomial can be defined as:

$$\begin{aligned}\Lambda(z) &= \prod_{j=1}^{M}(z - e^{2h\beta\phi_j})\prod_{j=1}^{M}(z - e^{-2h\beta\phi_j}) \\ &= \prod_{j=1}^{2M}(z - e^{2h\beta\rho_j}) = \sum_{l=0}^{2M}\lambda_l z^l,\end{aligned} \tag{4.2.4}$$

where

$$\rho_j = \begin{cases} \phi_j, & 1 \leq j \leq M \\ -\phi_{j-M}, & M < j \leq 2M, \end{cases} \tag{4.2.5}$$

and λ_l are the coefficients of the monomial terms in $\Lambda(z)$ with $\lambda_{2M} = 1$. Then we can derive

$$\begin{aligned}\sum_{l=0}^{2M}\lambda_l(\mathcal{S}_{K,(l+m)h}f)(\omega_0) &= \sum_{l=0}^{2M}\lambda_l e^{\beta h(l+m)(2x_0+h(l+m))}f(x_0 + h(l+m)) \\ &= \sum_{l=0}^{2M}\lambda_l e^{\beta h(l+m)(2x_0+h(l+m))}\left(\sum_{j=1}^{M}c_j e^{-\beta(\phi_j-(x_0+h(l+m)))^2}\right. \\ &\left.+\sum_{j=1}^{M}c_j e^{-\beta(\phi_j+(x_0+h(l+m)))^2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{2M} \lambda_l e^{\beta h(l+m)(2x_0+h(l+m))} \sum_{j=1}^{M} c_j e^{-\beta(\phi_j-(x_0+h(l+m)))^2} \\
&+ \sum_{l=0}^{2M} \lambda_l e^{\beta h(l+m)(2x_0+h(l+m))} \sum_{j=1}^{M} c_j e^{-\beta(\phi_j+(x_0+h(l+m)))^2} \\
&= \left(\sum_{j=1}^{M} c_j e^{-\beta(x_0+hm-\phi_j)^2} e^{\beta hm(2x_0+hm)} \right) \\
&\cdot \left(\sum_{l=0}^{2M} \lambda_l e^{-\beta(h^2l^2+2hl(x_0+mh-\phi_j))} e^{\beta hl(2x_0+h(l+2m))} \right) \\
&+ \left(\sum_{j=1}^{M} c_j e^{-\beta(x_0+hm+\phi_j)^2} e^{\beta hm(2\omega_0+hm)} \right) \\
&\cdot \left(\sum_{l=0}^{2M} \lambda_l e^{-\beta(h^2l^2+2hl(x_0+mh+\phi_j))} e^{\beta hl(2x_0+h(l+2m))} \right) \\
&= \left(\sum_{j=1}^{M} c_j e^{-\beta(\omega_0+hm-\phi_j)^2} e^{\beta hm(2x_0+hm)} \right) \underbrace{\left(\sum_{l=0}^{2M} \lambda_l e^{2\beta hl\phi_j} \right)}_{=0} \\
&+ \left(\sum_{j=1}^{M} c_j e^{-\beta(x_0+hm+\phi_j)^2} e^{\beta hm(2x_0+hm)} \right) \underbrace{\left(\sum_{l=0}^{2M} \lambda_l e^{-2\beta hl\phi_j} \right)}_{=0} = 0.
\end{aligned}
\tag{4.2.6}
$$

Thus, we obtain the following linear system

$$
\begin{aligned}
&\sum_{l=0}^{2M-1} \lambda_l e^{\beta h(l+m)(2x_0+h(l+m))} f(x_0+h(l+m)) \\
&\qquad = -e^{\beta h(m+2M)(2x_0+h(m+2M))} f(x_0+h(m+2M)) \quad m = 0, 1, ..., 2M-1.
\end{aligned}
\tag{4.2.7}
$$

Therefore, the vector of coefficients $\boldsymbol{\lambda} := (\lambda_0, ..., \lambda_{2M-1})^T$ can be obtained by solving the inhomogeneous system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{G}, \tag{4.2.8}$$

where $\boldsymbol{G} := \big((\mathcal{S}_{K,(M+m)h} f)(x_0)\big)_{m=0}^{2M-1}$, and $\boldsymbol{H} := \big((\mathcal{S}_{K,(l+m)h} f)(x_0)\big)_{l,m=0}^{2M-1,2M-1}$

is the Hankel matrix, which is invertible since,

$$
\begin{aligned}
\mathbf{H} :&= \left(K(x_0, h(l+m)) f(x_0 + h(l+m)) \right)_{l,m=0}^{2M-1} \\
&= \left(e^{\beta h(l+m)(2x_0+h(l+m))} \left[\sum_{j=1}^{M} c_j e^{-\beta(\phi_j-(x_0+h(l+m)))^2} + \sum_{j=1}^{M} c_j e^{-\beta(T_j+(x_0+h(l+m)))^2} \right] \right)_{l,m=0}^{2M-1} \\
&= \left(\sum_{j=1}^{M} c_j e^{\beta h(l+m)(2x_0+h(l+m))} e^{-\beta(\phi_j-(x_0+h(l+m)))^2} \right. \\
&\left. + \sum_{j=1}^{M} c_j e^{\beta h(l+m)(2x_0+h(l+m))} e^{-\beta(\phi_j+(x_0+h(l+m)))^2} \right)_{l,m=0}^{2M-1} \\
&= \left(\sum_{j=1}^{M} c_j e^{-\beta(\phi_j-x_0)^2} e^{h(l+m)\phi_j} + \sum_{j=1}^{M} c_j e^{-\beta(\phi_j+x_0)^2} e^{-h(l+m)\phi_j} \right)_{l,m=0}^{2M-1} \\
&= \mathbf{V}_h \mathrm{diag}(c_j e^{-\beta(\phi_j-x_0)^2} + c_j e^{-\beta(\phi_j+x_0)^2}) \mathbf{V}_h^T = \boldsymbol{V}_h \boldsymbol{D} \boldsymbol{V}_h^T
\end{aligned} \tag{4.2.9}
$$

with Vandermonde block matrix

$$
\mathbf{V}_h = \left(\begin{array}{c|c} \boldsymbol{A} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{B} \end{array} \right) \tag{4.2.10}
$$

where

$$
\boldsymbol{A} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{2\beta h\phi_1} & e^{2\beta h\phi_2} & \dots & e^{2\beta h\phi_M} \\ \vdots & \vdots & \dots & \vdots \\ e^{2(M-1)\beta h\phi_1} & e^{2(M-1)\beta h\phi_2} & \dots & e^{2(M-1)\beta h\phi_M} \end{bmatrix}
$$

$$
\boldsymbol{B} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{-2\beta h\phi_1} & e^{-2\beta h\phi_2} & \dots & e^{-2\beta h\phi_M} \\ \vdots & \vdots & \dots & \vdots \\ e^{-2(M-1)\beta h\phi_1} & e^{-2(M-1)\beta h\phi_2} & \dots & e^{-2(M-1)\beta h\phi_M} \end{bmatrix}
$$

and the diagonal block matrix

$$\mathbf{D} = \left[\begin{array}{cc|cc} c_1 e^{(\phi_1 - x_0)^2} & & & \\ & \ddots \; c_M e^{(\phi_M - x_0)^2} & & \mathbf{0} \\ \hline & \mathbf{0} & c_1 e^{(\phi_1 + x_0)^2} & \\ & & & \ddots \; c_M e^{(\phi_M + x_0)^2} \end{array}\right]. \tag{4.2.11}$$

Finally, the coefficients c_j of the expansion (4.2.1) can be computed by solving the following linear system:

$$f(x_0+lh) = \sum_{j=1}^{M} c_j e^{-\beta(\phi_j+x_0-lh)^2} + \sum_{j=1}^{M} c_j e^{-\beta(\phi_j+x_0+lh)^2}, \quad l = 0, ..., 4M-1. \tag{4.2.12}$$

Remark. *The parameters that we need to recover in the expansion* (4.2.1) *are M coefficients and M frequencies, but in the calculation we get $2M$ coefficients and $2M$ frequencies. As a result we have repeated coefficients c_j's, and pairs of opposite frequencies $(\phi_j, -\phi_j)$'s.*

Algorithm 5 Reconstruction of Mixed Expansions Using Generalized Shifted Operator (4.2.1).

Input:

- Number of terms M.
- Sampling values , $f(x_0 + lh), l = 0, ..., 4M - 1$.
- Choose sampling size h.

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$

where $\boldsymbol{H} := \big((\mathcal{S}_{K,(l+m)h} f)(x_0)\big)_{l,m=0}^{2M-1,2M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{2M-1}$ and $\boldsymbol{F} := \big((\mathcal{S}_{K,(2M+m)h} f)(x_0)\big)_{m=0}^{2M-1}$.

- Find all the zeros $z_j := e^{2h\beta\rho_j}, j = 1, ..., 2M$ of the polynomial

$$\Lambda(z) = \sum_{l=1}^{2M} \lambda_l z^l$$

and calculate ϕ_j from z_j for all $j = 1, \ldots, 2M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

$$f(x_0 + lh) = \sum_{j=1}^{M} c_j e^{-\beta(\phi_j + x_0 - lh)^2} + \sum_{j=1}^{M} c_j e^{-\beta(\phi_j + x_0 + lh)^2}, \quad l = 0, ..., 4M - 1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

4.3 Reconstruction of Cosine Expansion Using Short Time Fourier Sample

One of the most commonly used expansions , specially in practice, in signal processing is the cosine expansion. This expansion has been used to illustrate for example the Empirical Mode decomposition (EMD) experiments . In this section, we use a new technique to recover the parameters using Fourier data.

Let us consider our expansion in the form of

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x), \tag{4.3.1}$$

for non-zero complex parameters c_j and distinct real-valued frequencies ϕ_j , $j = 1, ..., M$ with $M \geq 1$.

In order to recover the parameters in the expansion f in (4.3.1) , we use the short time Fourier data. First, we use the representation of the short time Fourier transform (STFT), defined in (2.3.1) to derive the transform of f in the frequency domain.

Now, let us compute the STFT of the expansion (4.3.1)

$$
\begin{aligned}
STFT(\omega,\tau) &= \int_{-\infty}^{\infty}\left(\sum_{j=1}^{M} c_j\cos(\phi_j x)\right)w(x-\tau)e^{-i\omega x}dx \\
&= \int_{-\infty}^{\infty}\left(\sum_{j=1}^{M} c_j\left[\frac{1}{2}\left(e^{i\phi_j x}+e^{-i\phi_j x}\right)\right]\right)w(x-\tau)e^{-i\omega x}dx \\
&= \frac{1}{2}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{i\phi_j x}e^{-i\omega x}w(x-\tau)dx + \frac{1}{2}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{-i\phi_j x}e^{-i\omega x}w(x-\tau)dx \\
&= \frac{1}{2}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{i(\phi_j-\omega)x}w(x-\tau)dx + \frac{1}{2}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{-i(\phi_j+\omega)x}w(x-\tau)dx.
\end{aligned}
\tag{4.3.2}
$$

By using a Gaussian window defined in (4.1.3), we observe that

$$
\begin{aligned}
STFT(\omega,\tau) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2\sigma^2}+i(\phi_j-\omega)x}dx \\
&+ \frac{1}{2}\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2\sigma^2}-i(\phi_j+\omega)x}dx \\
&= \frac{1}{\sqrt{8\pi}}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}+i(\phi_j-\omega-\frac{i\tau}{\sigma^2})x-\frac{\tau^2}{2\sigma^2}}dx \\
&+ \frac{1}{\sqrt{8\pi}}\sum_{j=1}^{M} c_j\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}-i(\phi_j+\omega+\frac{i\tau}{\sigma^2})x-\frac{\tau^2}{2\sigma^2}}dx \\
&= \frac{1}{\sqrt{8\pi}}\sum_{j=1}^{M} c_j\sqrt{2\pi\sigma^2}e^{-2\sigma^2(\frac{\phi_j-\omega-i\tau/\sigma^2}{2})^2-\frac{\tau^2}{2\sigma^2}} \\
&+ \frac{1}{\sqrt{8\pi}}\sum_{j=1}^{M} c_j\sqrt{2\pi\sigma^2}e^{-2\sigma^2(\frac{\phi_j+\omega+i\tau/\sigma^2}{2})^2-\frac{\tau^2}{2\sigma^2}} \\
&= \frac{\sigma}{2}\sum_{j=1}^{M} c_je^{-2\sigma^2(\frac{\phi_j-\omega-i\tau/\sigma^2}{2})^2-\frac{\tau^2}{2\sigma^2}} + \frac{\sigma}{2}\sum_{j=1}^{M} c_je^{-2\sigma^2(\frac{\phi_j+\omega+i\tau/\sigma^2}{2})^2-\frac{\tau^2}{2\sigma^2}} \\
&= \sum_{j=1}^{M} \tilde{c}_je^{-2\sigma^2(\frac{\phi_j-\omega-i\tau/\sigma^2}{2})^2-\frac{\tau^2}{2\sigma^2}} + \sum_{j=1}^{M} \tilde{c}_je^{-2\sigma^2(\frac{\phi_j+\omega+i\tau/\sigma^2}{2})^2-\frac{\tau^2}{2\sigma^2}}
\end{aligned}
\tag{4.3.3}
$$

where $\tilde{c}_j = \frac{\sigma}{2}c_j$.

By setting $\tau = 0$ and $\sigma = 1$, we have

$$P(\omega) = STFT(\omega, 0) = \sum_{j=1}^{M} \tilde{c}_j e^{-\frac{1}{2}(\phi_j - \omega)^2} + \sum_{j=1}^{M} \tilde{c}_j e^{-\frac{1}{2}(\phi_j + \omega)^2}, \quad \omega \in \mathbb{R}. \tag{4.3.4}$$

This model is related to the model in (4.2.1) when $\beta = \frac{1}{2}$. In the same way as described in section (4.2), the coefficients c_j and the frequencies ϕ_j in the expansion (4.3.1) can be recovered by using $4M$ short time Fourier sampling values $P(\omega), \omega = 0, ..., 4M-1$.

Remark. *Similarly, the sine expansions of the form*

$$f(x) = \sum_{j=1}^{M} c_j \sin(\phi_j x) \tag{4.3.5}$$

can also be reconstructed by using the method described in section (4.3).

Algorithm 6 Reconstruction of Cosine Expansion Using Short Time Fourier Sample (4.3.1).

Input:

- Number of terms M.
- Sampling values , $P(\omega_0 + lh), l = 0, ..., 4M-1$ using (4.3.4).
- Choose sampling size h and arbitrary number ω_0.

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H\lambda} = -\boldsymbol{F},$$

where $\boldsymbol{H} := \left((\mathcal{S}_{K,(l+m)h}P)(\omega_0)\right)_{l,m=0}^{2M-1,2M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{2M-1}$ and $\boldsymbol{F} := \left((\mathcal{S}_{K,(2M+m)h}P)(\omega_0)\right)_{m=0}^{2M-1}$.

- Find all the zeros $z_j := e^{h\rho_j}, j = 1, ..., 2M$ of the polynomial

$$\Lambda(z) = \sum_{l=1}^{2M} \lambda_l z^l$$

and calculate ϕ_j from z_j for $j = 1, ..., 2M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

$$f(\omega_0 + lh) = \sum_{j=1}^{M} c_j e^{-\frac{1}{2}(\phi_j + \omega_0 - lh)^2} + \sum_{j=1}^{M} c_j e^{-\frac{1}{2}(\phi_j + \omega_0 + lh)^2}, \quad l = 0, ..., 4M - 1.$$

Output: ϕ_j and $c_j, j = 1, ..., M$.

4.4 Reconstruction of Non-stationary Signals with Quadratic Phase Functions Using Short Time Fourier Transform

In this section, we extend the cosine expansion to reconstruct a non-stationary signal of a special form. We use the same idea that we used in the previous section. For this end, we consider our signal in the following form

$$f(x) = \sum_{j=1}^{M} c_j \cos(x^2 + \phi_j x) \tag{4.4.1}$$

for non-zero complex parameters c_j and distinct real-valued frequencies ϕ_j , $j = 1, ..., M$ with $M \geq 1$.

First, we compute the STFT of the expansion (4.4.1) as follows:

$$\begin{aligned} STFT(\omega, \tau) &= \int_{-\infty}^{\infty} \left(\sum_{j=1}^{M} c_j \cos(x^2 + \phi_j x) \right) w(x - \tau) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} \left(\sum_{j=1}^{M} c_j \left[\frac{1}{2} \left(e^{i(x^2 + \phi_j x)} + e^{-i(x^2 + \phi_j x)} \right) \right] \right) w(x - \tau) e^{-i\omega x} dx \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2\sigma^2} + i(x^2 + \phi_j x) - i\omega x} dx \\ &+ \frac{1}{2} \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{2\sigma^2} - i(x^2 + \phi_j x) - i\omega x} dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{8\pi}} \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{-(\frac{1-2\sigma^2 i}{2\sigma^2})x^2+(i\phi_j-i\omega+\frac{\tau}{\sigma^2})x-\frac{\tau^2}{2\sigma^2}} dx \\
&+ \frac{1}{\sqrt{8\pi}} \sum_{j=1}^{M} c_j \int_{-\infty}^{\infty} e^{-(\frac{1+2\sigma^2 i}{2\sigma^2})x^2+(-i\phi_j-i\omega+\frac{\tau}{\sigma^2})x-\frac{\tau^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{8\pi}} \sum_{j=1}^{M} c_j \sqrt{2\pi} \sqrt{\frac{\sigma^2}{1-2\sigma^2 i}} e^{\frac{\sigma^2}{2(1-2\sigma^2 i)}(i\phi_j-i\omega-\frac{\tau}{\sigma^2})^2-\frac{\tau^2}{2\sigma^2}} \\
&+ \frac{1}{\sqrt{8\pi}} \sum_{j=1}^{M} c_j \sqrt{2\pi} \sqrt{\frac{\sigma^2}{1+2\sigma^2 i}} e^{\frac{\sigma^2}{2(1+2\sigma^2 i)}(-i\phi_j-i\omega+\frac{\tau}{\sigma^2})^2-\frac{\tau^2}{2\sigma^2}} \qquad (4.4.2)\\
&= \frac{1}{2} \sqrt{\frac{\sigma^2}{1-2\sigma^2 i}} \sum_{j=1}^{M} c_j e^{-\frac{\sigma^2}{2(1-2\sigma^2 i)}(\phi_j-\omega-i\frac{\tau}{\sigma^2})^2-\frac{\tau^2}{2\sigma^2}} \\
&+ \frac{1}{2} \sqrt{\frac{\sigma^2}{1+2\sigma^2 i}} \sum_{j=1}^{M} c_j e^{-\frac{\sigma^2}{2(1+2\sigma^2 i)}(\phi_j+\omega-i\frac{\tau}{\sigma^2})^2-\frac{\tau^2}{2\sigma^2}} \\
&= \sum_{j=1}^{M} \tilde{c}_j e^{-\frac{\sigma^2}{2(1-2\sigma^2 i)}(\phi_j-\omega-i\frac{\tau}{\sigma^2})^2-\frac{\tau^2}{2\sigma^2}} + \sum_{j=1}^{M} \tilde{\tilde{c}}_j e^{-\frac{\sigma^2}{2(1+2\sigma^2 i)}(\phi_j+\omega-i\frac{\tau}{\sigma^2})^2-\frac{\tau^2}{2\sigma^2}}
\end{aligned}
$$

By setting $\tau = 0$ and $\sigma = 1$, we have

$$
\begin{aligned}
P(\omega) = STFT(\omega, 0) &= \sum_{j=1}^{M} \tilde{c}_j e^{-\frac{1}{2(1-2i)}(\phi_j-\omega)^2} + \sum_{j=1}^{M} \tilde{\tilde{c}}_j e^{-\frac{1}{2(1+2i)}(\phi_j+\omega)^2} \\
&= \sum_{j=1}^{M} \tilde{c}_j e^{-\frac{1+2i}{10}(\phi_j-\omega)^2} + \sum_{j=1}^{M} \tilde{\tilde{c}}_j e^{-\frac{1-2i}{10}(\phi_j+\omega)^2}
\end{aligned} \qquad (4.4.3)
$$

where $\tilde{c}_j = \frac{1}{2}\sqrt{\frac{1-2i}{5}} c_j$ and $\tilde{\tilde{c}}_j = \frac{1}{2}\sqrt{\frac{1+2i}{5}} c_j$.

This model is related to the model in (4.2.1) when $\beta = \frac{1+2i}{10}$, and therefore the parameters c_j and ϕ_j can be reconstructed using $4M$ short time Fourier sampling values.

Algorithm 7 Reconstruction of Non-stationary Signals with Quadratic Phase Functions Using Short time Fourier Transform(4.4.1).

Input:

- Number of terms M.
- Sampling values , $P(\omega_0 + lh), l = 0, ..., 4M - 1$ using (4.4.3).
- Choose sampling size h and ω_0.

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F}$$

where $\boldsymbol{H} := \big((\mathcal{S}_{K,(l+m)h}P)(\omega_0)\big)_{l,m=0}^{2M-1,2M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{2M-1}$ and $\boldsymbol{F} := \big((\mathcal{S}_{K,(2M+m)h}P)(\omega_0)\big)_{m=0}^{2M-1}$.

- Find all the zeros $z_j := e^{h\rho_j}, j = 1, ..., 2M$ of the polynomial

$$\Lambda(z) = \sum_{l=1}^{2M} \lambda_l z^l$$

and calculate ϕ_j from z_j for all $j = 1, \ldots, 2M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

$$f(\omega_0+lh) = \sum_{j=1}^{M} c_j e^{-\frac{1+2i}{10}(\phi_j+\omega_0-lh)^2} + \sum_{j=1}^{M} c_j e^{-\frac{1-2i}{10}(\phi_j+\omega_0+lh)^2}, \quad l = 0, ..., 4M-1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

4.5 Reconstruction of Expansions Using Variable Substitutions

In this section, we study a general expansion that allow us to solve some special expansions. In this purpose, we reconstruct expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j (G(x))^{\phi_j} \tag{4.5.1}$$

where $G(x)$ is continuous and strictly monotonous function in $[a, b]$.

First, let us review the expansion discussed in [**12**] and [**17**] of the form

$$f(x) = \sum_{j=1}^{M} c_j x^{\phi_j} \tag{4.5.2}$$

with $c_j \in \mathbb{C} \backslash \{0\}$ and pairwise different $\phi_j \in \mathbb{C}$ that satisfy $\text{Im}\phi_j \in [-\frac{\pi}{h}, \frac{\pi}{h}]$.

Let us define the dilation operator

$$(\mathcal{D}_h f)(x) = f(e^h x).$$

Then we have

$$(\mathcal{D}_h (\cdot)^{\phi_j})(x_0) = (e^h x_0)^{\phi_j} = e^{h\phi_j} x_0^{\phi_j} \tag{4.5.3}$$

where $x_0 \in \mathbb{C} \backslash \{0\}$ is arbitrary.

We observe that the functions x^{ϕ_j} are the eigenfunctions of the dilation operator $\mathcal{D}_h$ with the eigenvalues $e^{h\phi_j}$. Then the expansion (4.5.2) can be reconstructed using $2M$ sampling values $f(e^{hl} x_0), l = 0, 1, ..., 2M - 1$.

Let us now consider the Prony polynomial

$$\Lambda(z) = \prod_{j=1}^{M} (z - e^{h\phi_j}) = \sum_{l=0}^{M} \lambda_l z^l \tag{4.5.4}$$

where $\lambda_l, l = 1, ..., M$ are the coefficients of the monomial representation (4.5.4) with $\lambda_M = 1$.

$$\begin{aligned} \sum_{l=0}^{M} \lambda_l f(e^{h(l+m)} x_0) &= \sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j e^{h(l+m)\phi_j} x_0^{\phi_j} \\ &= \sum_{j=1}^{M} c_j (e^{hm} x_0)^{\phi_j} \sum_{l=0}^{M} \lambda_l e^{h\phi_j l} \end{aligned}$$

$$= \sum_{j=1}^{M} c_j (e^{hm} x_0)^{\phi_j} \underbrace{\Lambda(e^{h\phi_j})}_{=0} = 0. \tag{4.5.5}$$

Therefore , we obtain the following linear system

$$\sum_{l=0}^{M-1} \lambda_l f(e^{h(l+m)} x_0) = -f(e^{h(M+m)} x_0). \tag{4.5.6}$$

By having λ_l and using the Prony polynomial, the coefficients c_j can be calculated by the following system

$$f(x_0 e^{hl}) = \sum_{j=1}^{M} c_j (x_0 e^{hl})^{\phi_j}, \quad l = 0, ..., 2M-1. \tag{4.5.7}$$

Now, by using substitution, we can recover the parameters of any expansion of the form (4.5.1).

Let $x = G(x)$, where $G(x)$ is an invertible function in $[a, b]$. Then the expansion (4.5.1) can be transferred to the model (4.5.2).

We note that

$$G(x) \longleftrightarrow x_0 e^{hl}, \quad \Longrightarrow x \longleftrightarrow G^{-1}(x_0 e^{hl}). \tag{4.5.8}$$

Therefore the expansion (4.5.1) can be reconstructed using also $2M$ sampling values $f(G^{-1}(x_0 e^{hl})), l = 0, ..., 2M-1$.

$$\begin{aligned} \sum_{l=0}^{M} \lambda_l f(G^{-1}(e^{h(k+m)} x_0)) &= \sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j \Big(G(G^{-1}(e^{h(l+m)} x_0)) \Big)^{\phi_j} \\ &= \sum_{j=1}^{M} c_j (e^{hm} x_0)^{\phi_j} \sum_{l=0}^{M} \lambda_l e^{h\phi_j l} \\ &= \sum_{j=1}^{M} c_j (e^{hm} x_0)^{\phi_j} \underbrace{\Lambda(e^{h\phi_j})}_{=0} = 0. \end{aligned} \tag{4.5.9}$$

Therefore , we obtain the following linear system

$$\sum_{l=0}^{M-1} \lambda_l f(G^{-1}(e^{h(l+m)} x_0)) = -f(G^{-1}(e^{h(M+m)} x_0)). \tag{4.5.10}$$

By having λ_l and using the Prony polynomial, the coefficients c_j can be calculated by the following system

$$f(G^{-1}(e^{hl}x_0)) = \sum_{j=1}^{M} c_j (G^{-1}(e^{hl}x_0))^{\phi_j}, \quad l = 0, ..., 2M-1. \tag{4.5.11}$$

Due to the flexibility of $G(x)$ in the model discussed above, we can use this structure to solve several expansion problems by selecting various inverse functions $G(x)$. We demonstrate this idea by the following two examples.

Example 4.5.1 Assume that $f(x)$ is a real-valued function of the form

$$f(x) = \sum_{j=1}^{M} c_j (\cos x)^{\phi_j}. \tag{4.5.12}$$

Here we take $G(x) = \cos x$ for $x \in [0, \pi]$, thus $G^{-1}(x) = \cos^{-1}(x)$. Then the expansion (4.5.12) can be reconstructed using $2M$ sampling values $f(\cos^{-1}(x_0 e^{hl})), l = 0, ..., 2M-1$.

The Prony polynomial has the form (4.5.4), therefore we have the following linear system

$$\begin{aligned}
\sum_{l=0}^{M} \lambda_l f(\cos^{-1}(e^{h(l+m)}x_0)) &= \sum_{k=0}^{M} \lambda_l \sum_{j=1}^{M} c_j \Bigg(\cos(\cos^{-1}(e^{h(l+m)}x_0)) \Bigg)^{\phi_j} \\
&= \sum_{j=1}^{M} c_j (e^{hm}x_0)^{\phi_j} \sum_{l=0}^{M} \lambda_l e^{h\phi_j l} \\
&= \sum_{j=1}^{M} c_j (e^{hm}x_0)^{\phi_j} \underbrace{\Lambda(e^{h\phi_j})}_{=0} = 0.
\end{aligned} \tag{4.5.13}$$

Therefore , we obtain the following linear system

$$\sum_{l=0}^{M-1} \lambda_l f(\cos^{-1}(e^{h(l+m)}x_0)) = -f(\cos^{-1}(e^{h(M+m)}x_0)). \tag{4.5.14}$$

By having λ_l and using the Prony polynomial, the coefficients c_j can be calculated by the following system

$$f(\cos^{-1}(e^{hl}x_0)) = \sum_{j=1}^{M} c_j (\cos(\cos^{-1}(e^{hl}x_0)))^{\phi_j}, \quad l = 0, ..., 2M-1. \tag{4.5.15}$$

Example 4.5.2 We consider $f(x)$ as a Gaussian expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j e^{\phi_j x^2}. \tag{4.5.16}$$

Here we take $G(x) = e^{x^2}$ for $x \in (1, \infty)$, thus $G^{-1}(x) = \sqrt{\ln x}$. Then the expansion (4.5.16) can be reconstructed using $2M$ sampling values $f(\sqrt{\ln(x_0 e^{hl})}), l = 0, ..., 2M-1$.

The Prony polynomial has the form (4.5.4) , therefore we have the following linear system

$$\begin{aligned}
\sum_{l=0}^{M} \lambda_l f(\sqrt{\ln(e^{h(l+m)} x_0)}) &= \sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j e^{\phi_j (\sqrt{\ln(e^{h(l+m)} x_0})^2} \\
&= \sum_{l=0}^{M} \lambda_l \sum_{j=1}^{M} c_j e^{\ln(e^{h(l+m)} x_0)^{\phi_j}} \\
&= \sum_{j=1}^{M} c_j (e^{hm} x_0)^{\phi_j} \sum_{l=0}^{M} \lambda_l e^{h\phi_j l} \\
&= \sum_{j=1}^{M} c_j (e^{hm} x_0)^{\phi_j} \underbrace{\Lambda(e^{h\phi_j})}_{=0} = 0.
\end{aligned} \tag{4.5.17}$$

Therefore , we obtain the following linear system

$$\sum_{l=0}^{M-1} \lambda_l f(\sqrt{\ln(e^{h(l+m)} x_0)}) = -f(\sqrt{\ln(e^{h(M+m)} x_0)}). \tag{4.5.18}$$

By having λ_l and using the Prony polynomial, the coefficients c_j can be calculated by the following system

$$f(\sqrt{\ln(e^{hl} x_0)}) = \sum_{k=0}^{M} c_j e^{\phi_j (\sqrt{\ln(e^{hl} x_0)})^2}, \quad l = 0, ..., 2M-1. \tag{4.5.19}$$

Algorithm 8 Reconstruction of Expansions Using Variable Substitutions (4.5.1).

Input:

- Number of terms M.
- Sampling values $f(G^{-1}(x_0 e^{hl})), l = 0, ..., 2M-1$.
- Choose sampling size h such that $h\phi_j \in (-\pi, \pi], \forall j \in \{1, ..., M\}$.

Calculation:

- Solve the inhomogeneous linear system
$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$
where $\boldsymbol{H} = (f(G^{-1}(e^{h(l+m)}x_0)))_{l,m=0}^{M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{M-1}$ and $\boldsymbol{F} = (f(G^{-1}(e^{h(M+m)}x_0)))_{m=0}^{M-1}$.
- Find all the zeros $z_j := e^{h\phi_j}, j = 1, ..., M$ of the polynomial
$$\Lambda(z) = \sum_{l=1}^{M} \lambda_l z^l$$
and calculate ϕ_j from z_j for $j = 1, ..., M$.
- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system
$$f(G^{-1}(e^{hl}x_0)) = \sum_{j=1}^{M} c_j (G^{-1}(e^{hl}x_0))^{\phi_j}, \quad l = 0, ..., 2M-1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

4.6 Reconstruction of Expansions Using Polynomials of Differential Operators

In this section, we use a different method to reconstruct some expansions that we reconstructed in the previous sections. We use a polynomial of the differential operator to allow us to apply the generalized Prony method. We first give the following definition[**2**]:

Definition 4.6.1. Let

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \tag{4.6.1}$$

be a polynomial of real variable x of degree n, where $a_0, a_1, ..., a_0$ are real constants. Then we define

$$P_n(D) = a_n D^n + a_{n-1}D^{n-1} + ... + a_1 D + a_0, \tag{4.6.2}$$

where $D = \frac{d}{dx}$, as a polynomial of differential operator D of degree n.

4.6.1 Prony Method for Exponential Sums Using the Polynomial of Differential Operators

Let us consider the polynomial of differential operator with

$$\mathcal{L} = P_n(D). \tag{4.6.3}$$

We observe that

$$\begin{aligned}
\mathcal{L}e^{\phi x} &= P_n(D)e^{\phi x} = P(\frac{d}{dx})e^{\phi x} \\
&= \left(a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + ... + a_1 D + a_0\right)e^{\phi x} \\
&= \left(a_n \phi^n + a_{n-1}\phi^{n-1} + ... + a_1\phi + a_0\right)e^{\phi x} \\
&= P_n(\phi)e^{\phi x}.
\end{aligned} \tag{4.6.4}$$

Then the operator $\mathcal{L}$ defined in (4.6.3) possesses the set $\{P_n(\phi), n \in \mathbb{N}\}$ of the different eigenvalues with corresponding eigenfunctions $e^{x\phi}$. We want to reconstruct the sparse sum of exponentials using the operator $\mathcal{L}$ of the form

$$f(x) = \sum_{j=1}^{M} c_j e^{x\phi_j} \tag{4.6.5}$$

The generalized Prony method can be applied to reconstruct the expansion (4.6.5) using the operator $\mathcal{L}$ with , $F(f) := f(x_0)$ and $2M$ sampling value as follow:

$$\mathcal{L}^0 f = \sum_{j=1}^{M} c_j (P_n(\phi_j))^0 e^{(\phi_j x_0)}$$

$$\mathcal{L}^1 f = \sum_{j=1}^{M} c_j (P_n(\phi_j))^1 e^{(\phi_j x_0)}$$

$$\begin{aligned}
&\mathcal{L}^2 f = \sum_{j=1}^{M} c_j (P_n(\phi_j))^2 e^{(\phi_j x_0)} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\mathcal{L}^{2M-1} f = \sum_{j=1}^{M} c_j (P_n(\phi_j))^{2M-1} e^{(\phi_j x_0)}.
\end{aligned} \tag{4.6.6}$$

Let us now consider the Prony polynomial

$$\Lambda(z) = \prod_{j=1}^{M} (z - P_n(\phi_j)) = \sum_{k=0}^{M} \lambda_k z^k \tag{4.6.7}$$

where $\lambda_k, k = 1, ..., M$ are the coefficients of the monomial representation (4.6.7) with $\lambda_M = 1$.

$$\begin{aligned}
\sum_{k=0}^{M} \lambda_k F(\mathcal{L}^{k+m} f) &= \sum_{k=0}^{M} \lambda_k F\left(\sum_{j=1}^{M} c_j (P_n(\phi_j))^{k+m} e^{\phi_j x_0} \right) \\
&= \sum_{j=1}^{M} c_j (P_n(\phi_j))^m \left(\sum_{k=0}^{M} \lambda_k (P_n(\phi_j))^k \right) F(e^{\phi_j x_0}) \\
&= \sum_{j=1}^{M} c_j (P_n(\phi_j))^m \underbrace{\Lambda(P_n(\phi_j))}_{=0} F(e^{\phi_j x_0}) = 0, \quad k = 0, ..., M-1.
\end{aligned} \tag{4.6.8}$$

Thus, we obtain the following linear system

$$\sum_{k=0}^{M-1} \lambda_k F(\mathcal{L}^{k+m} f) = -F(\mathcal{L}^{M+m} f), \quad m = 0, 1, ..., M-1. \tag{4.6.9}$$

By having the coefficients of Prony polynomial $\lambda_k, k = 0, ..., M$, then we can compute the coefficients $c_j, j = 1, ..., M$ of the expansion (4.6.5) by solving the overdetermined linear system

$$F(\mathcal{L}^k f)(x) = \sum_{j=1}^{M} c_j F\left((P_n(\phi_j))^k e^{x_0 \phi_j} \right), \quad k = 0, ..., 2M-1. \tag{4.6.10}$$

4.6.2 Reconstruction of Cosine Expansions Using Polynomials of Differential Operators

Now let us consider the polynomial of the differential operator D^2 as

$$\mathcal{L} = P_n(D^2). \tag{4.6.11}$$

Observe that

$$
\begin{aligned}
\mathcal{L}\cos(\phi x) &= P_n(D^2)\cos(\phi x) \\
&= \Big(a_n(D^2)^n + a_{n-1}(D^2)^{n-1} + ... + a_1 D^2 + a_0\Big)\cos(\phi x) \\
&= \Big(a_n(-\phi^2)^n + a_{n-1}(-\phi^2)^{n-1} + ... + a_1(-\phi^2) + a_0\Big)\cos(\phi x) \\
&= P_n(-\phi^2)\cos(\phi x).
\end{aligned}
\tag{4.6.12}
$$

Then the operator $\mathcal{L}$ defined in (4.6.11) possesses the set $\{P_n(-\phi^2), n \in \mathbb{N}\}$ of the different eigenvalues with corresponding eigenfunctions $\cos(\phi x)$. We want to reconstruct the cosine expansion using the operator $\mathcal{L}$ on the expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x). \tag{4.6.13}$$

The generalized Prony method can be applied to reconstruct the expansion (4.6.13) using the operator $\mathcal{L}$ with , $F(f) := f(x_0)$ and $2M$ sampling values as follow

$$
\begin{aligned}
\mathcal{L}^0 f &= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^0 \cos(\phi_j x_0) \\
\mathcal{L}^1 f &= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^1 \cos(\phi_j x_0) \\
\mathcal{L}^2 f &= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^2 \cos(\phi_j x_0) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathcal{L}^{2M-1} f &= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^{2M-1} \cos(\phi_j x_0).
\end{aligned}
\tag{4.6.14}
$$

Let us now consider the Prony polynomial

$$\Lambda(z) = \prod_{j=1}^{M}(z - P_n(-\phi_j^2)) = \sum_{k=0}^{M} \lambda_k z^k \tag{4.6.15}$$

where $\lambda_k, k = 1, ..., M$ are the coefficients of the monomial representation (4.6.15) with $\lambda_M = 1$.

$$\begin{aligned}\sum_{k=0}^{M}\lambda_k F(\mathcal{L}^{k+m}f) &= \sum_{k=0}^{M}\lambda_k F\Bigg(\sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^{k+m}\cos(\phi_j x_0)\Bigg)\\ &= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^{m}\Bigg(\sum_{k=0}^{M}\lambda_k (P_n(-\phi_j^2))^{k}\Bigg)F(\cos(\phi_j x_0))\\ &= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^{m}\underbrace{\Lambda(P_n(-\phi_j^2))}_{=0}F(\cos(\phi_j x_0)) = 0, \quad k = 0, ..., M-1.\end{aligned} \tag{4.6.16}$$

Thus, we obtain the following linear system

$$\sum_{k=0}^{M-1}\lambda_k F(\mathcal{L}^{k+m}f) = -F(\mathcal{L}^{M+m}f), \quad m = 0, 1, ..., 2M-1 \tag{4.6.17}$$

By having the coefficients of Prony polynomial $\lambda_k, k = 0, ..., M$, then we can compute the coefficients $c_j, j = 1, ..., M$ of the expansion (4.6.13) by solving the overdetermined linear system

$$F(\mathcal{L}^k f) = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^k \cos(\phi_j x_0), \quad k = 0, ..., 2M-1, \tag{4.6.18}$$

Remark. *Similarly, we can reconstruct the Sine expansion*

$$f(x) = \sum_{j=1}^{M} c_j \sin(\phi_j x) \tag{4.6.19}$$

by using

$$\mathcal{L}\sin(\phi x) = P_n(D^2)\sin(\phi x) = P_n(-\phi^2)\sin(\phi x). \tag{4.6.20}$$

Algorithm 9 Reconstruction of Cosine Expansions Using the Polynomial of Differential Operator (4.6.13).

Input:

- Number of terms M.
- Polynomial $P_n(x)$.
- $2M$ sampling values using (4.6.14).

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$

where $\boldsymbol{H} = (F(\mathcal{L}^{k+m} f))_{l,m=0}^{M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_k)_{k=0}^{M-1}$ and $\boldsymbol{F} = (F(\mathcal{L}^{M+m} f))_{m=0}^{M-1}$.

- Find all the zeros $z_j := P_n(-\phi_j^2), j = 1, ..., M$ of the polynomial

$$\Lambda(z) = \sum_{k=1}^{M} \lambda_k z^k$$

and calculate ϕ_j from z_j for $j = 1, ..., M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

$$F(\mathcal{L}^k f) = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^k \cos(\phi_j x_0), \quad k = 0, ..., 2M - 1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

4.7 Reconstruction of Gaussian Expansions Using Differential Operators

The Gaussian expansion can be also reconstructed by using certain differential operator. We consider the Gaussian expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j e^{\phi_j x^2}. \tag{4.7.1}$$

Let us define our differential operator as

$$\mathcal{D} = \frac{1}{x} . \frac{d}{dx}.$$

Then we have

$$\mathcal{D}e^{\phi_j x^2} = \frac{1}{x}.\frac{d}{dx}e^{\phi_j x^2} = 2\phi_j e^{\phi_j x^2}. \tag{4.7.2}$$

The Gaussian functions $e^{\phi_j x^2}$ are the eigenfunctions of the differential operator $\mathcal{D}$ with the eigenvalues $2\phi_j$. Therefore, the expansion (4.7.1) can be reconstructed using $2M$ sampling values $F(\mathcal{D}^k f), k = 0, ..., 2M-1$.

The Prony polynomial can be defined as

$$\Lambda(z) = \prod_{j=1}^{M}(z - 2\phi_j) = \sum_{k=0}^{M} \lambda_k z^k. \tag{4.7.3}$$

Thus

$$\begin{aligned}\sum_{k=0}^{M} \lambda_k F(\mathcal{D}^{k+m} f)(x_0) &= \sum_{k=0}^{M} \lambda_k F\left(\sum_{j=1}^{M} c_j (2\phi_j)^{k+m} e^{\phi_j x_0^2}\right) \\ &= \sum_{j=1}^{M} c_j (2\phi_j)^m \left(\sum_{k=0}^{M} \lambda_k (2\phi_j)^k\right) F(e^{\phi_j x_0}) \\ &= \sum_{j=1}^{M} c_j (2\phi_j)^m \underbrace{\Lambda(2\phi_j)}_{=0} F(e^{\phi_j x_0}) = 0.\end{aligned} \tag{4.7.4}$$

By having the coefficients of Prony polynomial $\lambda_k, k = 0, ..., M$, then we can compute the coefficients $c_j, j = 1, ..., M$ of the expansion (4.7.1) by solving the overdetermined linear system

$$F(\mathcal{D}^k f) = \sum_{j=1}^{M} c_j (2\phi_j)^k e^{\phi_j x_0}, \quad k = 0, ..., 2M-1. \tag{4.7.5}$$

4.8 Reconstruction Expansions Using Combination of Shift and Dilation Operators

There are some expansions that need a special operator for reconstruction. Let us consider the expansion of the following form

$$f(x) = \sum_{j=1}^{M} c_j e^{\frac{\phi_j}{x}}, \quad x \neq 0. \tag{4.8.1}$$

Now, we define our new operator as

$$\mathcal{S}_h f(x) = f\left(\frac{xh}{x+h}\right). \tag{4.8.2}$$

Then we have

$$(\mathcal{S}_h e^{\phi(\frac{1}{\cdot})})(x) = e^{\phi(\frac{1}{\frac{xh}{x+h}})} = e^{\phi(\frac{x+h}{xh})} = e^{\frac{\phi}{h}} e^{\frac{\phi}{x}}. \tag{4.8.3}$$

We observe that the functions $e^{\frac{\phi_j}{x}}$ are the eigenfunctions of the differential operator $\mathcal{S}_h$ with the eigenvalues $e^{\frac{\phi}{h}}$. Therefore, the expansion (4.8.1) can be reconstructed from $2M$ sampling values $f(\frac{x_0 h}{kx_0+h}), k = 0, ..., 2M-1$, where $x_0 \in \mathbb{R}\backslash\{0\}$ is an arbitrary real number.

The Prony polynomial can be defined as

$$\Lambda(z) = \prod_{j=1}^{M}(z - e^{\frac{\phi_j}{h}}) = \sum_{k=0}^{M} \lambda_k z^k. \tag{4.8.4}$$

Observe that

$$\begin{aligned} \sum_{k=0}^{M} \lambda_k f\left(\frac{x_0 h}{x_0(k+m)+h}\right) &= \sum_{k=0}^{M} \lambda_k \sum_{j=1}^{M} e^{\phi_j(\frac{k+m}{h}+\frac{1}{x_0})} \\ &= \sum_{j=1}^{M} c_j e^{\phi_j(\frac{mx_0+h}{x_0 h})} \sum_{k=0}^{M} \lambda_k e^{\frac{\phi_j k}{h}} \\ &= \sum_{j=1}^{M} c_j e^{\phi_j(\frac{mx_0+h}{x_0 h})} \underbrace{\Lambda(e^{\frac{\phi_j}{h}})}_{=0} = 0, \quad m = 0, ..., M-1. \end{aligned} \tag{4.8.5}$$

Thus, we obtain the following linear system

$$\sum_{k=0}^{M-1} \lambda_k f\left(\frac{x_0 h}{x_0(k+m)+h}\right) = -f\left(\frac{x_0 h}{x_0(k+M)+h}\right). \tag{4.8.6}$$

As usual, solving the system (4.8.6) provides us the coefficients λ_k of the Prony polynomial $\Lambda(z)$ and therefore all the zeros can be extracted.

Finally, the coefficients c_j of the expansion (4.8.1) can be obtained by solving the linear system

$$f\left(\frac{x_0 h}{kx_0+h}\right) = \sum_{j=1}^{M} c_j e^{\frac{\phi_j(kx_0+h)}{x_0 h}}, \quad k = 0, ..., 2M-1. \tag{4.8.7}$$

Algorithm 10 Reconstruction Expansions of the form (4.8.1) Using Combination of Shift and Dilation Operators.

Input:

- Number of terms M.
- Sampling values $f(\frac{x_0 h}{kx_0+h}), k = 0, ..., 2M-1$.
- Choose sampling size h such that $\frac{\phi_j}{h} \in (-\pi, \pi], \forall j \in \{1, ..., M\}$. **Calculation:**
- Solve the inhomogeneous linear system

$$\boldsymbol{H\lambda} = -\boldsymbol{F},$$

where $\boldsymbol{H} = \big(f(\frac{x_0 h}{x_0(k+m)+h})\big)_{l,m=0}^{M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_k)_{k=0}^{M-1}$ and $\boldsymbol{F} = \big(f(\frac{x_0 h}{x_0(k+M)+h})\big)_{k=0}^{M-1}$.

- Find all the zeros $z_j := e^{\frac{\phi_j}{h}}, j = 1, ..., M$ of the polynomial

$$\Lambda(z) = \sum_{k=1}^{M} \lambda_l z^k$$

and calculate ϕ_j from z_j for $j = 1, ..., M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

$$f\left(\frac{x_0 h}{kx_0 + h}\right) = \sum_{j=1}^{M} c_j e^{\frac{\phi_j (kx_0+h)}{x_0 h}}, \quad k = 0, ..., 2M-1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

4.9 Numerical Experiments

In this section we proceed to illustrate our methods in the previous sections to recover the parameters with some simple examples.

Example 4.9.1. Our first test is to recover the parameters of the signal with three components

$$f(x) = \sum_{j=1}^{3} c_j e^{ix\phi_j}, \tag{4.9.1}$$

where the parameters of (4.9.1) are given in the following table

	$j=1$	$j=2$	$j=3$
c_j	3.5000	-2.8000	5.7500
ϕ_j	0.8590	2.5590	2.7860

Table 4.1: Parameters of the function $f(x)$ in (4.9.1) .

By computing the STFT in (4.1.5) to the signal in (4.9.1), we have

$$STFT(\omega, 0) = c_1 e^{-\frac{1}{2}(\phi_1-\omega)^2} + c_2 e^{-\frac{1}{2}(\phi_2-\omega)^2} + c_3 e^{-\frac{1}{2}(\phi_3-\omega)^2}. \tag{4.9.2}$$

The Table (4.2) shows the absolute reconstruction error $|c_j - c_j^*|$ and $|\phi_j - \phi_j^*|$ where c_j^* and ϕ_j^* are reconstructed parameters and frequencies respectively.

j	c_j	ϕ_j	$\lvert\phi_j - \phi_j^*\rvert$	$\lvert\phi_j - \phi_j^*\rvert$
1	3.5000	-2.5590	$8.4377.10^{-15}$	$6.6613.10^{-16}$
2	-2.8000	0.8590	$1.0627.10^{-11}$	$4.2011.10^{-13}$
3	5.7500	2.7860	$1.0617.10^{-11}$	$2.2249.10^{-13}$

Table 4.2: Parameters of the function $f(x)$ in (4.9.1) and approximate errors using 6 short time Fourier sampling values with $h = 0.5$.

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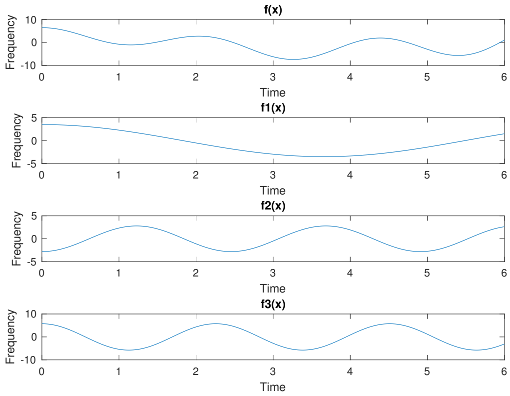

Figure 4.1: The signal $f(x)$ in (4.9.1) and the three components $f_1(x)$, $f_2(x)$, and $f_3(x)$.

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Example 4.9.2. We test our method on signal of the form (4.3.1) with three components

$$f(x) = \sum_{j=1}^{3} c_j \cos(\phi_j x), \tag{4.9.3}$$

where the parameters of (4.9.3) are given in the following table

	$j=1$	$j=2$	$j=3$
c_j	0.5000	0.2500	1.000
ϕ_j	1.0000	3.0000	4.0000

Table 4.3: Parameters of the function $f(x)$ in (4.9.3).

By computing STFT in (4.3.4) to the signal in (4.9.3), we obtain

$$\begin{aligned} STFT(\omega, 0) = c_1\big(e^{-\frac{1}{2}(\phi_1-\omega)^2} + e^{-\frac{1}{2}(\phi_1+\omega)^2}\big) + c_2\big(e^{-\frac{1}{2}(\phi_2-\omega)^2} + e^{-\frac{1}{2}(\phi_2+\omega)^2}\big) \\ + c_3\big(e^{-\frac{1}{2}(\phi_3-\omega)^2} + e^{-\frac{1}{2}(\phi_3+\omega)^2}\big). \end{aligned} \tag{4.9.4}$$

Similarly, as example (4.9.2) the error can be shown in the table (4.4)

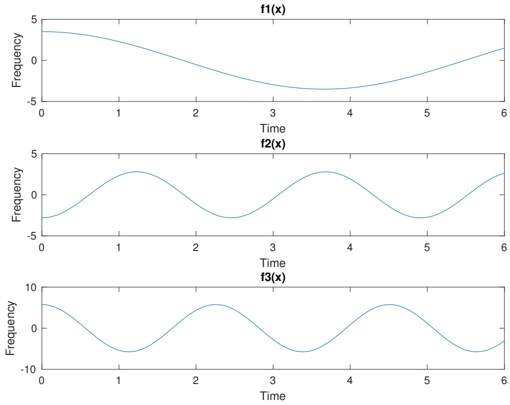

Figure 4.2: Components of the signal $f(x)$ in (4.9.1) using short time Fourier sampling values.

j	c_j	ϕ_j	$\lvert c_j - c_j^* \rvert$	$\lvert \phi_j - \phi_j^* \rvert$
1	0.5000	1.0000	$3.7970.10^{-12}$	$5.2824.10^{-13}$
2	0.2500	3.0000	$5.0987.10^{-14}$	$4.5652.10^{-13}$
3	1.0000	4.0000	$5.8065.10^{-14}$	$1.4211.10^{-14}$

Table 4.4: Parameters of the function $f(x)$ in (4.9.3) and approximate errors using 12 sampling values with $h = 0.5$.

Now, we reconstruct the parameters c_j and ϕ_j, $j = 1, 2, 3$ in example (4.9.3) using the polynomial of differential operator shown in (4.6.2) with $P_2(x) = 3x^2 - 2x + 12$, then we have the following result

j	c_j	ϕ_j	$\lvert c_j - c_j^* \rvert$	$\lvert \phi_j - \phi_j^* \rvert$
1	0.5000	1.0000	$1.4539.10^{-09}$	$4.8850.10^{-14}$
2	0.2500	3.0000	$8.1046.10^{-15}$	$5.9064.10^{-14}$
3	1.0000	4.0000	$4.6629.10^{-15}$	$8.8818.10^{-16}$

Table 4.5: Parameters of the function $f(x)$ in (4.9.3) and approximate errors using 12 sampling values as in (4.6.14).

Example 4.9.3. We consider the recovery of the expansion

$$f(x) = \sum_{j=1}^{M} c_j (G(x))^{\phi_j}$$

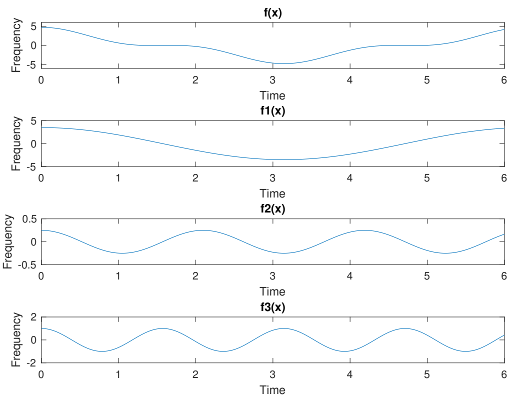

Figure 4.3: The signal $f(x)$ in (4.9.3) and the three components $f_1(x)$, $f_2(x)$, and $f_3(x)$.

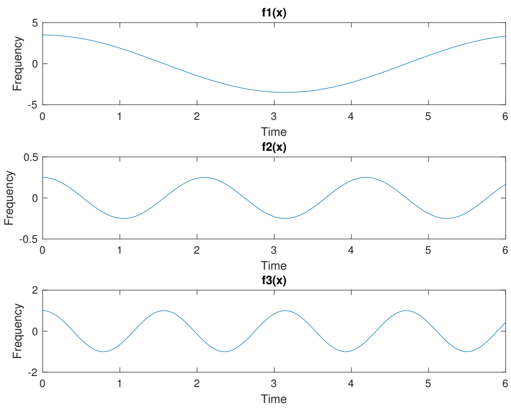

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Figure 4.4: Components of the signal $f(x)$ in (4.9.3) using short time Fourier sampling values.

with $M = 5$ and $G(x) = e^{x^2}$ i.e

$$f(x) = \sum_{j=1}^{5} c_j e^{\phi_j x^2}, \tag{4.9.5}$$

where the parameters c_j and ϕ_j, $j = 1, ..., 5$ are given by

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
c_j	0.5456	-1.7865	2.4542	-0.2139	4.6754
ϕ_j	0.2556	0.8654	2.5463	3.5000	4.3643

Table 4.6: Parameters of the function $f(x)$ in (4.9.5) .

j	c_j	ϕ_j	$\lvert c_j - c_j^*\rvert$	$\lvert \phi_j - \phi_j^*\rvert$
1	0.5456	0.2556	$8.3373.10^{-08}$	$9.0436.10^{-08}$
2	-1.7865	0.8654	$1.4495.10^{-07}$	$3.7487.10^{-08}$
3	2.4542	2.5463	$5.0929.10^{-09}$	$1.8464.10^{-09}$
4	-0.2139	3.5000	$1.5335.10^{-09}$	$1.9291.10^{-09}$
5	4.6754	4.3643	$2.8431.10^{-11}$	$8.3222.10^{-13}$

Table 4.7: Parameters of the function $f(x)$ in (4.9.5) and approximate errors using 10 sampling values with $h = 0.5$.

Example 4.9.4. We consider the recovery of the expansion

$$f(x) = \sum_{j=1}^{6} e^{\frac{\phi_j}{x}}, \tag{4.9.6}$$

where the parameters c_j and ϕ_j, $j = 1, ...6$ are given in the table (4.8)

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
c_j	1.5641	4.2342	-0.2342	3.6300	1.7864	5.2020
ϕ_j	0.1213	1.3242	2.2376	3.6811	3.8942	4.4321

Table 4.8: Parameters of the function $f(x)$ in (4.9.6).

Using the method explained in (4.8), we have

j	c_j	ϕ_j	$\lvert c_j - c_j^*\rvert$	$\lvert \phi_j - \phi_j^*\rvert$
1	1.5641	0.1213	$2.8920.10^{-05}$	$1.0497.10^{-05}$
2	4.2342	1.3242	$7.7826e.10^{-06}$	$9.4657.10^{-06}$
3	-0.2342	2.2376	$3.4489.10^{-05}$	$6.2593.10^{-05}$
4	3.6300	3.6811	$4.3056.10^{-05}$	$1.8754.10^{-06}$
5	1.7864	3.8942	$4.4888.10^{-05}$	$2.2554.10^{-06}$
6	5.2020	4.4321	$2.2087.10^{-07}$	$3.7128.10^{-09}$

Table 4.9: Parameters of the function $f(x)$ in (4.9.6) and approximate errors using 12 sampling values with $h = 3$.

Chapter 5

Expansions with Two Models

In signal processing it is possible to have signals that contain different models. In the previous chapter, we discussed the reconstruction expansions of signals that follow one model. In this chapter, we study the recovery methods for linear combinations of two different models. In sections (5.1) and (5.2) we consider the signals in combination of sine and cosine models using two different methods. In section (5.3), we use even and odd properties of the functions to reconstruct the signals of two models.

5.1 Reconstruction of Linear Combinations of Sine and Cosine Expansions Using the Polynomials of Differential Operators

Let us consider the expansion with the following form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) + \sum_{j=1}^{M} c_j \sin(\phi_j x) = \sum_{j=1}^{M} c_j \big(\cos(\phi_j x) + \sin(\phi_j x)\big). \quad (5.1.1)$$

Let P_n be a polynomial and $\mathcal{L}$ be the polynomial of the differential operator D^2 defined in (4.6.2), that is,

$$\mathcal{L} = P_n(D^2). \quad (5.1.2)$$

Observe that

$$\begin{aligned} \mathcal{L}(\cos(\phi x) + \sin(\phi x)) &= P_n(D^2)(\cos(\phi x) + \sin(\phi x)) \\ &= P_n(-\phi^2)(\cos(\phi x) + \sin(\phi x)). \end{aligned} \quad (5.1.3)$$

Then the operator $\mathcal{L}$ defined in (5.1.2) possesses the set $\{P_n(-\phi_j^2), n \in \mathbb{N}\}$ of the different eigenvalues with corresponding eigenfunctions $\cos(\phi_j x) + \sin(\phi_j x)$.

The generalized Prony method can applied to reconstruct the expansion (5.1.1)using the operator $\mathcal{L}$ with , $F(f) := f(x_0)$ and $2M$ sampling values as follow

$$\begin{aligned}
&\mathcal{L}^0 f = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^0 \big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big) \\
&\mathcal{L}^1 f = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^1 \big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big) \\
&\mathcal{L}^2 f = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^2 \big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\mathcal{L}^{2M-1} f = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^{2M-1} \big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big).
\end{aligned} \tag{5.1.4}$$

Let us now consider the Prony polynomial

$$\Lambda(z) = \prod_{j=1}^{M} (z - P_n(-\phi_j^2)) = \sum_{k=0}^{M} \lambda_k z^k, \tag{5.1.5}$$

where $\lambda_k, k = 1, ..., M$ are the coefficients of the monomial representation (5.1.5) with $\lambda_M = 1$.

We observe that

$$\begin{aligned}
\sum_{k=0}^{M} \lambda_k F(\mathcal{L}^{k+m} f) &= \sum_{k=0}^{M} \lambda_k F\Bigg(\sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^{k+m} (\cos(\phi_j x_0) + \sin(\phi_j x_0)\Bigg) \\
&= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^m \Bigg(\sum_{k=0}^{M} \lambda_k (P_n(-\phi_j^2))^k \Bigg) F\big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big) \\
&= \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^m \underbrace{\Lambda(P_n(-\phi_j^2))}_{=0} F\big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big) = 0 \\
&\quad , k = 0, ..., M-1.
\end{aligned} \tag{5.1.6}$$

Thus, we obtain the following linear system

$$\sum_{k=0}^{M-1} \lambda_k F(\mathcal{L}^{k+m} f) = -F(\mathcal{L}^{M+m} f), \quad m = 0, 1, ..., M-1. \tag{5.1.7}$$

By having the coefficients of Prony polynomial $\lambda_k, k = 0, ..., M$, then we can compute the coefficients $c_j, j = 1, ..., M$ of the expansion (5.1.1) by solving the overdetermined linear system

$$F(\mathcal{L}^k f) = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^k (\cos(\phi_j x_0) + \sin(\phi_j x_0), \quad k = 0, ..., 2M-1. \tag{5.1.8}$$

The coefficient matrix $\mathbf{H} := \big(F(\mathcal{L}^{k+m})\big)_{k=0,m=0}^{M-1,M-1}$ is an invertible Hankel matrix, since it can be written as

$$\mathbf{H} = \mathbf{V}_M(P_n(-\phi_j^2))\mathrm{diag}(c_j)\mathrm{diag}\big(F(\cos(x_0\phi_j) + \sin(x_0\phi_j)\big)\mathbf{V}_M(P_n(-\phi_j^2))^T \tag{5.1.9}$$

with $\mathbf{V}_M := \big(P_n(-\phi_j^2)^k\big)_{k=0}^{M-1}$.

Algorithm 11 Reconstruction of Linear Combinations of Sine and Cosine Expansions Using the Polynomials of Differential Operators (5.1.1).

Input:

- Number of terms M.
- Polynomial $P_n(x)$.
- $2M$ sampling values using (5.1.4).

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$

  where $\boldsymbol{H} = (F(\mathcal{L}^{k+m} f))_{l,m=0}^{M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_k)_{k=0}^{M-1}$ and $\boldsymbol{F} = (F(\mathcal{L}^{M+m} f))_{m=0}^{M-1}$.

- Find all the zeros $z_j := P_n(-\phi_j^2), j = 1, ..., M$ of the polynomial

$$\Lambda(z) = \sum_{k=1}^{M} \lambda_k z^k,$$

  and calculate ϕ_j from z_j for $j = 1, ..., M$.

- Find the unknown coefficients $c_j, j = 1, ..., M$ by solving the linear system

---

$$F(\mathcal{L}^k f) = \sum_{j=1}^{M} c_j (P_n(-\phi_j^2))^k \big(\cos(\phi_j x_0) + \sin(\phi_j x_0)\big), \quad k = 0, ..., 2M-1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

5.2 Reconstruction of Linear Combinations of Sine and Cosine Expansions Using Chebyshev Polynomials

Let us consider expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) + \sum_{j=1}^{M} d_j \sin(\beta_j x). \tag{5.2.1}$$

By using the symmetric shift operator defined in (2.6.1), we have

$$\begin{aligned}(\mathcal{S}_{h,-h}) \cos(\phi x) &= \frac{1}{2}\bigg(\cos(\phi(x+h)) + \cos(\phi(x-h))\bigg) \\ &= \cos(\phi h)\cos(\phi x).\end{aligned} \tag{5.2.2}$$

Similarly, we apply the symmetric shift operator on $\sin(\beta x)$, and get

$$\begin{aligned}(\mathcal{S}_{h,-h}) \sin(\beta x) &= \frac{1}{2}\bigg(\sin(\beta(x+h)) + \sin(\beta(x-h))\bigg) \\ &= \cos(\beta h)\sin(\beta x).\end{aligned} \tag{5.2.3}$$

Let us now define the Prony polynomial as follows:

$$\Lambda(z) = \prod_{j=1}^{M} (z - \cos(h\phi_j)) \prod_{j=1}^{M} (z - \cos(h\beta_j)). \tag{5.2.4}$$

Therefore the polynomial (5.2.4) can be written in terms of Chebyshev polynomial as

$$\Lambda(z) = \sum_{k=0}^{2M} \lambda_k \boldsymbol{T}_k(z), \tag{5.2.5}$$

where $\mathbf{T}_k(z) := \cos(k\cos^{-1}(z))$. We note that the leading coefficient of the Chebyshev polynomial $\mathbf{T}_k(z)$ is 2^{k-1}, then by the definition of the Prony polynomial Λ, we have $\lambda_M = 2^{1-M}$.

Observe that

$$\begin{aligned}
&\sum_{k=0}^{2M} \lambda_k \Big(\mathcal{S}_{kh,-kh}) \mathcal{S}_{mh,-mh} f(x_0) \Big) \\
&= \frac{1}{2} \sum_{k=0}^{2M} \lambda_k \Big(f(x_0 + (m+k)h) + f(x_0 - (m+k)h) + f(x_0 + (m-k)h) + f(x_0 - (m-k)h)) \Big) \\
&= \frac{1}{2} \sum_{k=0}^{2M} \lambda_k \Big[\sum_{j=1}^{M} 2c_j \Big(\cos(\phi_j(x_0 + mh)) + \cos(\phi_j(x_0 - mh)) \Big) \cos(\phi_j kh) \Big] \\
&+ \frac{1}{2} \sum_{k=0}^{2M} \lambda_k \Big[\sum_{j=1}^{M} 2d_j \Big(\sin(\beta_j(x_0 + mh)) + \sin((\beta_j(x_0 - mh)) \Big) \cos(\beta_j kh) \Big] \\
&= \sum_{j=1}^{M} c_j \Big(\cos(\phi_j(x_0 + mh)) + \cos(\phi_j(x_0 - mh)) \Big) \sum_{k=0}^{2M} \lambda_k \cos(\phi_j kh) \\
&+ \sum_{j=1}^{M} d_j \Big(\sin(\beta_j(x_0 + mh)) + \sin((\beta_j(x_0 - mh)) \Big) \sum_{k=0}^{2M} \lambda_k \cos \beta_j(kh) \\
&= \sum_{j=1}^{M} c_j \Big(\cos(\phi_j(x_0 + mh)) + \cos(\phi_j(x_0 - mh)) \Big) \underbrace{\sum_{k=0}^{2M} \lambda_k \boldsymbol{T}_k(\cos(\phi_j h))}_{=0} \\
&+ \sum_{j=1}^{M} d_j \Big(\sin(\beta_j(x_0 + mh)) + \sin((\beta_j(x_0 - mh)) \Big) \underbrace{\sum_{k=0}^{2M} \lambda_k \boldsymbol{T}_k(\cos(\beta_j h))}_{=0} = 0.
\end{aligned} \tag{5.2.6}$$

Here we choose $x_0 \neq 0$, more precisely we have $\beta_j x_0 \neq (2k+1)\frac{\pi}{2}$ and $\sin(\beta_j x_0) \neq 0$ for $k \in \mathbf{Z}$ and $j = 1, ..., M$. Therefore, we need to take all values of $\mathcal{S}_{lh,-lh}\mathcal{S}_{mh,-mh} f(x_0)$. Then we have the following linear system

$$
\begin{aligned}
\sum_{k=0}^{2M-1} \lambda_k \Big(f(x_0 + (m+k)h) + f(x_0 - (m+k)h) + f(x_0 + (m-k)h) + f(x_0 - (m-k)h) \Big) \\
= -2^{1-M} \Big(f(x_0 + (2M+k)h) + f(x_0 - (2M+k)h) \\
+ f(x_0 + (2M-k)h) + f(x_0 - (2M-k)h) \Big).
\end{aligned}
\tag{5.2.7}
$$

As in the classical Prony method, the coefficient matrix can be factorized as two Vandermonde Block matrices and diagonal block matrix

$$
\begin{aligned}
\mathbf{H} :&= \Big(f(x_0 + (m+k)h) + f(x_0 - (m+k)h) + f(x_0 + (m-k)h) + f(x_0 - (m-k)h) \Big)_{m,k=0}^{2M-1} \\
&= 4 \Big(\sum_{j=1}^{M} c_j \cos(\phi_j x_0) \cos(\phi_j m h) \cos(\phi_j k h) + \sum_{j=1}^{M} d_j \sin(\beta_j x_0) \cos(\beta_j m h) \cos(\beta_j k h) \Big)_{m,k=0}^{2M-1} \\
&= 4 \boldsymbol{V}_h \mathrm{diag}\big(c_j \cos(\phi_j x_0) + d_j \sin(\beta_j x_0) \big)_{j=1}^{M} \boldsymbol{V}^T = 4 \boldsymbol{V}_h \boldsymbol{D} \boldsymbol{V}_h^T
\end{aligned}
\tag{5.2.8}
$$

the Vandermonde Block matrix can be written as

$$
\boldsymbol{V}_h = \left(\begin{array}{c|c} \boldsymbol{A} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{B} \end{array} \right), \tag{5.2.9}
$$

where

$$
\boldsymbol{A} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ T_1(\cos \phi_1 h) & T_1(\cos \phi_2 h) & \dots & T_1(\cos \phi_M h) \\ \vdots & \vdots & \dots & \vdots \\ T_{2M-1}(\cos \phi_1 h) & T_{2M-1}(\cos \phi_2 h) & \dots & T_{2M-1}(\cos \phi_M h) \end{bmatrix} \tag{5.2.10}
$$

and

$$
\boldsymbol{B} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ T_1(\cos \beta_1 h) & T_1(\cos \beta_2 h) & \dots & T_1(\cos \beta_M h) \\ \vdots & \vdots & \dots & \vdots \\ T_{2M-1}(\cos \beta_1 h) & T_{2M-1}(\cos \beta_2 h) & \dots & T_{2M-1}(\cos \beta_M h) \end{bmatrix} \tag{5.2.11}
$$

and diagonal block matrix can be written as

$$\boldsymbol{D} = \left[\begin{array}{ccc|ccc} c_1\cos(\phi_1 x_0) & & & & & \\ & \ddots & & & \mathbf{0} & \\ & & c_M\cos(\phi_M x_0) & & & \\ \hline & & & d_1\sin(\beta_1 x_0) & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & d_M\sin(\beta_M x_0) \end{array}\right]. \tag{5.2.12}$$

By having λ_k and using the Prony polynomial, the coefficients c_j can be calculated by the following system

$$f(x_0+hl) = \sum_{j=1}^{M} c_j\cos(\phi_j(x_0+hl)) + \sum_{j=1}^{M} d_j\sin(\beta_j(x_0+hl)), \quad l = 0,...,4M-1. \tag{5.2.13}$$

Algorithm 12 Reconstruction of Linear Combination of Sine and Cosine Expansions Using Chebyshev Polynomial (5.2.1).

Input:

- Number of terms M.
- $4M$ sampling values.

Calculation:

- Solve the inhomogeneous linear system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{F},$$

where $\boldsymbol{H} = \Big((f(x_0+(m+k)h)+f(x_0-(m+k)h)+f(x_0+(m-k)h)+f(x_0-(m-k)h))\Big)_{l,m=0}^{2M-1}$ is a Hankel matrix, $\boldsymbol{\lambda} = (\lambda_k)_{k=0}^{2M-1}$ and $\boldsymbol{F} = \Big(2^{1-M}\Big(f(x_0+(2M+k)h)+f(x_0-(2M+k)h)+f(x_0+(2M-k)h)+f(x_0-(2M-k)h)\Big)\Big)_{m=0}^{2M-1}$.

- Find all the zeros z_j for $j = 1,...,2M$ of the polynomial

$$\Lambda(z) = \sum_{k=0}^{2M} \lambda_k \boldsymbol{T}_k(z).$$

- Find all frequencies ϕ_j and β_j for $j = 1, ..., M$ using

$$\left\{\frac{\cos^{-1}(z_1)}{h}, \frac{\cos^{-1}(z_2)}{h}, \ldots, \frac{\cos^{-1}(z_{2M})}{h}\right\}.$$

- Find the unknown coefficients c_j and $d_j, j = 1, ..., M$ by solving the linear system

$$f(x_0+hl) = \sum_{j=1}^{M} c_j \cos(\phi_j(x_0+hl)) + \sum_{j=1}^{M} d_j \sin(\beta_j(x_0+hl)), \quad l = 0, ..., 4M-1.$$

Output:

- ϕ_j and $c_j, j = 1, ..., M$.

5.3 Reconstruction of Expansions of Linear Combinations of Two Models Using Odd and Even Properties

In this section, we study signals that have odd and even models in the form

$$f(x) = \sum_{j=1}^{M} c_j G(x) + \sum_{j=1}^{M} d_j Q(x), \tag{5.3.1}$$

where $G(x)$ is an odd signal and $Q(x)$ is an even signal.

Let us consider expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) + \sum_{j=1}^{M} d_j \sin(\beta_j x). \tag{5.3.2}$$

We want to recover the parameters c_j, d_j, ϕ_j,and β_j.

Note that our expansion is the combination of even and odd models. First we calculate $f(-x)$

$$f(-x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) - \sum_{j=1}^{M} d_j \sin(\beta_j x). \tag{5.3.3}$$

By adding (5.3.2) and (5.3.3) we can get

$$P(x) = \frac{f(x) + f(-x)}{2} = \sum_{j=1}^{M} c_j \cos(\phi_j x). \tag{5.3.4}$$

The coefficients c_j and the frequencies ϕ_j in (5.3.4) can be recovered by $2M - 1$ sampling values $P(lh), l = 0, 1, ..., 2M - 1$. see section (3.5).

By subtracting (5.3.2) and (5.3.3) we can get

$$Q(x) = \frac{f(x) - f(-x)}{2} = \sum_{j=1}^{M} d_j \sin(\beta_j x). \tag{5.3.5}$$

Similarly, the coefficients d_j and the frequencies β_j can be reconstructed by using $Q(lh), l = 0, 1, ..., 2M - 1$, see also section (3.5).

Remark. *Similarly, the signal models in the form*

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) + \sum_{j=1}^{M} d_j \tan(\phi_j x) \tag{5.3.6}$$

and

$$f(x) = \sum_{j=1}^{M} c_j \cos(\phi_j x) + \sum_{j=1}^{M} d_j x e^{\beta_j x} \tag{5.3.7}$$

can be also reconstructed since they have odd and even models structures.

5.4 Numerical Experiments

Example 5.4.1. We test our method by given the signal with three components

$$f(x) = \sum_{j=1}^{3} c_j \big(\cos(\phi_j x) + \sin(\phi_j x) \big) \tag{5.4.1}$$

where the parameters of (5.4.1) are given in the following table

	$j = 1$	$j = 2$	$j = 3$
c_j	-2.4321	0.4532	8.3250
ϕ_j	2.7876	5.6654	6.3041

Table 5.1: Parameters of the function $f(x)$ in (5.4.1)

The table (5.1) shows the absolute reconstruction error $|c_j - c_j^*|$ and $|\phi_j - \phi_j^*|$ where c_j^* and ϕ_j^* are reconstructed parameters and frequencies respectively.

j	c_j	ϕ_j	$\lvert\phi_j - \phi_j^*\rvert$	$\lvert\phi_j - \phi_j^*\rvert$
1	-2.4321	2.7876	$1.8158.10^{-09}$	$2.2204.10^{-15}$
2	0.4532	5.6654	$4.1389.10^{-13}$	$2.9843.10^{-13}$
3	8.3250	6.30041	$1.7231.10^{-13}$	$3.5527.10^{-15}$

Table 5.2: Parameters of the function $f(x)$ in (5.4.1) and approximate errors using 6 sampling values with $x_0 = 1$ and $P_n(x) = 2x^2 + 3x - 1$.

Example 5.4.2. Our second example is to test the method in the form (5.2.1)

$$f(x) = \sum_{j=1}^{2} c_j \cos(\phi_j x) + \sum_{j=1}^{2} d_j \sin(\beta_j x) \tag{5.4.2}$$

where the parameters of (5.4.2) are given in the following table

j	c_j	d_j	ϕ_j	β_j
1	-2.6570	-0.5643	0.7865	4.2132
2	3.5643	4.4321	3.5432	5.1232

Table 5.3: Parameters of the function $f(x)$ in (5.4.2)

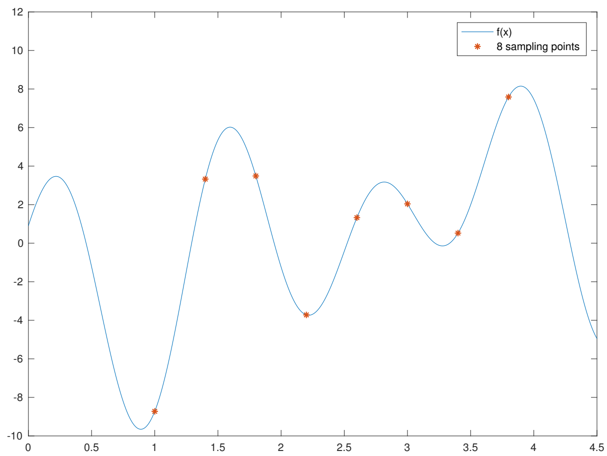

Figure 5.1: The signal $f(x)$ in (5.4.2) with 8 sampling values.

The table (5.4) shows the absolute reconstruction error $|c_j - c_j^*|$, $|d_j - d_j^*|$, $|\phi_j - \phi_j^*|$ and $|\beta_j - \beta_j^*|$ where c_j^*, d_j^*, ϕ_j^* and β_j^* are reconstructed parameters and frequencies respectively.

j	$\lvert c_j - c_j^* \rvert$	$\lvert d_j - d_j^* \rvert$	$\lvert \phi_j - \phi_j^* \rvert$	$\lvert \beta_j - \beta_j^* \rvert$
1	$3.1486.10^{-13}$	$1.5464.10^{-12}$	$3.1086.10^{-15}$	$6.4659.10^{-13}$
2	$1.4211.10^{-12}$	$7.1765.10^{-13}$	$4.1744.10^{-14}$	$1.7764.10^{-14}$

Table 5.4: Approximate errors using 8 sampling values with $x_0 = 1$ and $h = 0.4$ of the function (5.4.2).

Chapter 6

Fixing Corrupted Sampling Values

In this chapter, we consider a situation that some of the sampling values are *incorrect*, which could happen in real-world applications. For the *Signal Recovery Problem* in terms of sparse expansions, when the sampling values are selected, it could happen that some of the values are corrupted due to certain problems in data acquisition or data transmission. If we use any corrupted sampling value in the Prony method or generalized Prony methods, the outcome would be unpredictable, because a small error in a sampling value would result in some dramatic change in computation.

In order to recover the original signals, we can only rely on those correct sampling values. However, we may not know what sampling values are *correct*. Since those *incorrect* sampling values cannot participate in any computation step, we need some extra sampling values to overcome our loss on those corrupted sampling values. Thus for this problem, we require over-sampled data points that could provide us sufficient information to find a solution. How to detect and fix those incorrect sampling values? In this chapter, we will use several methods to study this problem.

6.1 Reduction Method

Problem A (Ideal Case): To recover a sparse expansion system with M terms as follows,

$$f(x) = \sum_{j=1}^{M} c_j e^{\phi_j x}, \tag{6.1.1}$$

we are given a sequence of *equidistance* sampling values $S := \{f(nh)\}_{n=1}^{2M+k}$ for some fixed step size h. It is also known that there are k sampling values in S that are *incorrect*, but we do not know their locations. How to recover this system?

First we consider the simplest case for *Problem A*, that is when $k = 1$.

Problem A1: To recover a sparse expansion system with M terms as follows,

$$f(x) = \sum_{j=1}^{M} c_j e^{\phi_j x},$$

we have are given a sequence of *equidistance* sampling values $S := \{f(nh)\}_{n=1}^{2M+1}$ for some fixed step size h. It is also known that there is exact 1 sampling value in S that is incorrect, but we do not know its location. How to recover this system?

Solution: To simplify the notation slightly, we denote S as $\{f_n\}_{n=1}^{2M+1}$. Based on the standard *Prony Method*, we need to solve the following system:

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f_{M+1} & f_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} & f_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f_{M+1} & f_{M+2} & f_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \\ 1 \end{bmatrix} = 0, \tag{6.1.2}$$

where $p_1, p_2, \ldots, p_M$ are *unknowns* (that are the coefficients of the Prony polynomial without the leading coefficient), and one of the coefficients in $\{f_1, f_2, \ldots, f_{2M+1}\}$ is *incorrect*, but we do not know which one. Since the incorrect coefficient could be any number in $\{f_1, f_2, \ldots, f_{2M+1}\}$, there are total $2M + 1$ different cases to be considered. Here let us use τ to denote the subscript that corresponds to the incorrect coefficient.

Step 1: Claim that there exists one *correct equation* for a fixed τ.

Here when we say the *correct equation*, we mean that all the coefficients of this equation are correct, that is, there subscripts are different from τ. We consider the following cases:

- *Case S1.1*: When τ is an *even* integer.

In this case, we simply select the last equation, that is,

$$f_1p_1 + f_3p_2 + f_5p_3 + \cdots + f_{2M-1}p_M + f_{2M+1} = 0. \tag{6.1.3}$$

- *Case S1.2*: When τ is an *odd* integer.

Since τ is an odd integer, we can write it as: $\tau = 2\sigma + 1$ with $0 \le \sigma \le M$. Now we rewrite the last equation that contains $f_{2\sigma+1}$ as follows,

$$f_1p_1 + f_3p_2 + \cdots + f_{2\sigma+1}p_{\sigma+1} + \cdots + f_{2M-1}p_M + f_{2M+1} = 0. \tag{6.1.4}$$

Next we select the $(\sigma + 1)$-th equation from (6.1.2), and get

$$f_{\sigma+1}p_1 + \cdots + f_{2\sigma+1}p_{\sigma+1} + \cdots + f_{M+\sigma}p_M + f_{M+\sigma+1} = 0. \tag{6.1.5}$$

Now we just subtract (6.1.4) from (6.1.5), and get

$$\begin{aligned}(f_{\sigma+1} - f_1)p_1 + \cdots + (f_{2\sigma} - f_{2\sigma-1})p_\sigma + (f_{2\sigma+2} - f_{2\sigma+3})p_{\sigma+2} + \cdots \\ + (f_{M+\sigma+1} - f_{2M+1}) = 0.\end{aligned} \tag{6.1.6}$$

Now we write a combined version for the *correct equation* as follows,

$$\xi_1p_1 + \xi_2p_2 + \cdots + \xi_Mp_M + \xi_{M+1} = 0, \tag{6.1.7}$$

where

$$\xi_i = \begin{cases} f_{\sigma+i} - f_{2i-1}, & \text{for } \tau \text{ even and } i = 1, 2, \ldots, M+1 \\ f_{2i-1}, & \text{for } \tau \text{ odd and } i = 1, 2, \ldots, M+1. \end{cases} \tag{6.1.8}$$

Step 2: Express one of the variables in terms of other variables.

From (6.1.7) in *Step 1*, first we assume that $\xi_M \neq 0$ (we will consider the case that $\xi_M = 0$ later using another method). From (6.1.7), we get

$$p_M = -\frac{\xi_{M+1}}{\xi_M} - \frac{\xi_1}{\xi_M}p_1 - \cdots - \frac{\xi_{M-1}}{\xi_M}p_{M-1}. \tag{6.1.9}$$

Step 3: Express those contaminated terms in terms of good terms.

First we consider the case that $\tau = M + 1$. In this case, every given equation (except the last one if M is odd) has a term containing f_{M+1}. We move those terms to the left-hand side of their corresponding equations, and place all the *good* terms (that means their coefficients are all correct) at the right-hand side.

For the first equation, we write it as

$$f_{M+1} = -f_1 p_1 - f_2 p_2 - \cdots - f_M p_M. \tag{6.1.10}$$

For the last such equation, we write it as

$$f_{M+1} p_1 = -f_{2M+1} - f_{M+2} p_2 - \cdots - f_{2M} p_M. \tag{6.1.11}$$

For the equations from (2) to (M), we have a general form for them,

$$\begin{aligned} f_{M+1} p_i = -f_{2M-i+2} - f_{M-i+2} p_1 - \cdots - f_M p_{i-1} - f_{M+2} p_{i+1} \\ - \cdots - f_{2M-i+1} p_M, \quad \text{for } 2 \le i \le M. \end{aligned} \tag{6.1.12}$$

Since the given value for f_{M+1} is incorrect, we need to find its correct value. Thus we treat f_{M+1} as an unknown in equations (6.1.10) – (6.1.12). Therefore our goal is to find the values of $(M+1)$ unknowns $p_1, \ldots, p_M, f_{M+1}$ from the system (6.1.10) – (6.1.12) with the help of (6.1.9).

Step 4: Reduction: Write p_{M-1} in terms of $p_1, p_2, \ldots, p_{M-2}$.

We start with a simplified version of (6.1.9), that is,

$$p_M = \eta_0^M + \eta_1^M p_1 + \cdots + \eta_{M-1}^M p_{M-1}, \tag{6.1.13}$$

where

$$\eta_i^M = -\frac{\xi_i}{\xi_M} \quad \text{for } 1 \le i \le M-1, \quad \text{and} \quad \eta_0^M = -\frac{\xi_{M+1}}{\xi_M}. \tag{6.1.14}$$

With (6.1.13), we can eliminate the p_M-term at the right-hand side of (6.1.10), (6.1.11) and (6.1.12) in this way:

After the substitution, (6.1.10) becomes

$$f_{M+1} = \alpha_0^0 + \alpha_1^0 p_1 + \alpha_2^0 p_2 + \cdots + \alpha_{M-1}^0 p_{M-1}, \tag{6.1.15}$$

where

$$\alpha_j^0 = -f_j - f_M \eta_j^M, \quad \text{for} \quad 1 \le j \le M-1; \quad \text{and} \quad \alpha_0^0 = -f_M \eta_0^M. \tag{6.1.16}$$

Similarly, (6.1.11) becomes

$$f_{M+1} p_1 = \alpha_0^1 + \alpha_1^1 p_1 + \alpha_2^1 p_2 + \cdots + \alpha_{M-1}^1 p_{M-1}, \tag{6.1.17}$$

where

$$\alpha_j^1 = -f_{M+j} - f_{2M}\eta_j^M, \quad \text{for} \quad 2 \le j \le M-1, \tag{6.1.18}$$

and

$$\alpha_0^1 = -f_{2M+1} - f_{2M}\eta_0^M \quad \text{and} \quad \alpha_1^1 = -f_{2M}\eta_1^M. \tag{6.1.19}$$

For (6.1.12), we have

$$f_{M+1}p_i = \alpha_0^i + \alpha_1^i p_1 + \alpha_2^i p_2 + \cdots + \alpha_{M-1}^i p_{M-1}, \tag{6.1.20}$$

where

$$\alpha_j^i = -f_{M-i+j+1} - f_{2M-i+1}\,\eta_j^M, \quad \text{for} \quad 1 \le j\,(j \ne i) \le M-1, \tag{6.1.21}$$

and

$$\alpha_0^i = -f_{2M-i+2} - f_{2M-i+1}\,\eta_0^M \quad \text{and} \quad \alpha_i^i = -f_{2M-i+1}\,\eta_i^M. \tag{6.1.22}$$

Thus we can summarize the three cases in (6.1.15), (6.1.17) and (6.1.20) in the following general form:

$$f_{M+1}p_i = \alpha_0^i p_0 + \alpha_1^i p_1 + \alpha_2^i p_2 + \cdots + \alpha_{M-1}^i p_{M-1}, \quad \text{for } 0 \le i \le M. \tag{6.1.23}$$

where we assume that

$$p_0 = 1. \tag{6.1.24}$$

Now we can start the reduction process by multiplying (6.1.13) both sides by f_{M+1}, and get

$$f_{M+1}p_M = \eta_0^M f_{M+1}p_0 + \eta_1^M f_{M+1}p_1 + \cdots + \eta_{M-1}^M f_{M+1}p_{M-1}. \tag{6.1.25}$$

With the help of (6.1.23), the right-hand side of (6.1.25) can be represented as

$$\begin{aligned}\sum_{j=0}^{M-1}\eta_j^M f_{M+1}p_j &= \sum_{j=0}^{M-1}\eta_j^M \begin{bmatrix}\alpha_0^j & \alpha_1^j & \cdots & \alpha_{M-1}^j\end{bmatrix}\begin{bmatrix}p_0\\ p_1\\ \vdots\\ p_{M-1}\end{bmatrix}\\ &= \begin{bmatrix}\sum_{j=0}^{M-1}\eta_j^M\alpha_0^j, & \sum_{j=0}^{M-1}\eta_j^M\alpha_1^j, & \cdots & \sum_{j=0}^{M-1}\eta_j^M\alpha_{M-1}^j\end{bmatrix}\begin{bmatrix}p_0\\ p_1\\ \vdots\\ p_{M-1}\end{bmatrix}.\end{aligned} \tag{6.1.26}$$

From (6.1.23) for $i = M$, we can get the left-hand side of (6.1.25) as

$$f_{M+1}p_M = \alpha_0^M p_0 + \alpha_1^M p_1 + \alpha_2^M p_2 + \cdots + \alpha_{M-1}^M p_{M-1}. \tag{6.1.27}$$

Combining (6.1.25)-(6.1.27), we get

$$\beta_0^M p_0 + \beta_1^M p_1 + \beta_2^M p_2 + \cdots + \beta_{M-1}^M p_{M-1} = 0, \tag{6.1.28}$$

where

$$\beta_j^M = \alpha_j^M - \sum_{k=0}^{M-1} \eta_k^M \alpha_j^k, \qquad \text{for } 0 \le j \le M-1. \tag{6.1.29}$$

If the coefficients β_j^M's are not all zero, our reduction step can get to the next level. Let us consider the case for which we have $\beta_{M-1}^M \neq 0$. (For the case that $\beta_{M-1}^M = 0$, we have another method for it.) Then we can write (6.1.28) as follows,

$$p_{M-1} = \eta_0^{M-1} + \eta_1^{M-1} p_1 + \cdots + \eta_{M-2}^{M-1} p_{M-2}, \tag{6.1.30}$$

where

$$\eta_j^{M-1} = -\frac{\beta_j^M}{\beta_{M-1}^M} \qquad \text{for } 0 \le j \le M-2. \tag{6.1.31}$$

Similarly, we can derive the following equation from (6.1.30)

$$p_{M-2} = \eta_0^{M-2} + \eta_1^{M-2} p_1 + \cdots + \eta_{M-3}^{M-2} p_{M-3}, \tag{6.1.32}$$

where

$$\eta_j^{M-2} = -\frac{\gamma_j^{M-1}}{\gamma_{M-2}^{M-1}} \qquad \text{for } 0 \le j \le M-3, \tag{6.1.33}$$

with

$$\gamma_j^{M-1} = \alpha_j^{M-1} - \sum_{k=0}^{M-2} \eta_k^{M-1} \alpha_j^k, \qquad \text{for } 0 \le j \le M-2 \tag{6.1.34}$$

and assume that $\gamma_{M-2}^{M-1} \neq 0$. (For the case that $\gamma_{M-2}^{M-1} = 0$, we leave it to another more general method.)

If we can keep doing the above reduction process again and again, eventually we will get

$$p_1 = \eta_0^1, \tag{6.1.35}$$

and we will get all the values of $p_2, \ldots, p_M$, and f_{M+1} from the inverse process of the reduction. $\square$

Remark. *In the reduction method above, there are some steps where we assume certain coefficients are non-zero. But it is possible that some of those coefficients could be zero sometimes. If that case happens, we will use the following more general method.*

6.2 Determinant Method

In the *reduction method* we discussed above, there are a few cases for which some intermediate coefficients could be zero that would cause some complexity in our method. For this situation, we have another more general method that can cover all the above cases.

This is a *determinant-based* method that is guaranteed to work all the time for $k = 1$. However the above *reduction method* still has its own value in computation due to two reasons: First it works most of the time; second, it is a *linear method* and the computation is relatively easy.

Observe that the equation (6.1.2) can be viewed as a homogeneous equation which always has a *non-trivial* solution, because the solution vector $[p_1, \ldots, p_M, 1]^T$ can never be zero. Thus the $(M+2) \times (M+1)$ matrix at the left-hand side of (6.1.2) cannot be of full rank, that is, the determinant of any $(M+1) \times (M+1)$ submatrix of this $(M+2) \times (M+1)$ matrix must be zero. This observation can help us to find that f_τ which is incorrect. Based on the problem description, we notice that a solution must exist, and we just need to find the correct f_τ value that corresponds to the solution.

More specifically, we consider $2M+1$ cases where $\tau = 1, 2, \ldots, 2M+1$. For each case, we just remove one of the $(M+2)$ equations and get an $(M+1) \times (M+1)$ submatrix, and make sure that the remaining equations contain the variable f_τ. Thus when we calculate the determinant of this $(M+1) \times (M+1)$ submatrix, it is a univariate polynomial of f_τ. We find the real roots of this polynomial for possible true values for f_τ. Among those finitely many real values of f_τ, there must be one that corresponds to the solution. We locate it as follows.

Assume that f_τ^0 is one of the possible correct values for f_τ. We write the equation (6.1.2) in the following form by skipping the last equation,

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M \\ f_2 & f_3 & f_4 & \cdots & f_{M+1} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} \\ & \vdots & & \cdots & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} \\ f_{M+1} & f_{M+2} & f_{M+3} & \cdots & f_{2M} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \end{bmatrix} = - \begin{bmatrix} f_{M+1} \\ f_{M+2} \\ \vdots \\ f_{2M} \\ f_{2M+1} \end{bmatrix}, \tag{6.2.1}$$

where the value of f_τ^0 is used to replace the given value of f_τ at the left-hand side of (6.2.1). We split the equation (6.2.1) into the following two equations,

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M \\ f_2 & f_3 & f_4 & \cdots & f_{M+1} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} \\ & \vdots & & \cdots & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \end{bmatrix} = - \begin{bmatrix} f_{M+1} \\ f_{M+2} \\ \vdots \\ f_{2M} \end{bmatrix}, \tag{6.2.2}$$

and

$$\begin{bmatrix} f_2 & f_3 & f_4 & \cdots & f_{M+1} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} \\ & \vdots & & \cdots & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} \\ f_{M+1} & f_{M+2} & f_{M+3} & \cdots & f_{2M} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \end{bmatrix} = - \begin{bmatrix} f_{M+2} \\ \vdots \\ f_{2M} \\ f_{2M+1} \end{bmatrix}. \tag{6.2.3}$$

If f_τ^0 is the correct one, then the following two Hankel matrices

$$A_1 := \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M \\ f_2 & f_3 & f_4 & \cdots & f_{M+1} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} \\ & \vdots & & \cdots & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} \end{bmatrix} \tag{6.2.4}$$

and

$$A_2 := \begin{bmatrix} f_2 & f_3 & f_4 & \cdots & f_{M+1} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} \\ & \vdots & & \cdots & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} \\ f_{M+1} & f_{M+2} & f_{M+3} & \cdots & f_{2M} \end{bmatrix} \tag{6.2.5}$$

must be invertible. Thus we can get the solution for (6.2.1) in the following two expressions,

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \end{bmatrix} = -A_1^{-1} \begin{bmatrix} f_{M+1} \\ f_{M+2} \\ \vdots \\ f_{2M} \end{bmatrix} \tag{6.2.6}$$

and

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \end{bmatrix} = -A_2^{-1} \begin{bmatrix} f_{M+2} \\ \vdots \\ f_{2M} \\ f_{2M+1} \end{bmatrix}. \tag{6.2.7}$$

Only when

$$A_1^{-1} \begin{bmatrix} f_{M+1} \\ f_{M+2} \\ \vdots \\ f_{2M} \end{bmatrix} = A_2^{-1} \begin{bmatrix} f_{M+2} \\ \vdots \\ f_{2M} \\ f_{2M+1} \end{bmatrix}, \tag{6.2.8}$$

we get a potential solution for $p_1, \ldots, p_M$. But we still need to verify the last equation in (6.1.2) using the values for $p_1, \ldots, p_M$ from (6.2.6), that is,

$$f_1 p_1 + f_3 p_2 + f_5 p_3 + \cdots + f_{2M-1} p_M + f_{2M+1} = 0. \tag{6.2.9}$$

If it is *incorrect*, we will go to the next possible correct value for f_τ and repeat the above verification step until we find the right value for f_τ. If it is correct, then we recover $f(x)$ using these $p_1, p_2, \ldots, p_M$. After that, we evaluate $f(nh)$ for $n = 1, 2, \ldots, 2M + 1$, and see if they are *compatible* with the given sampling values. Here the *compatibility* requirement means that the two sets of sampling values are the same except for the corrupted value f_τ. If yes, we find the solution. If not, we will go to the next possible correct value for f_τ and repeat the above verification step until we find the right value for f_τ .

Remark. *If $k > 1$, then the* determinant method *would result in a system of equations with polynomials of k variables, which is very hard to solve in general. If we can find a method that could reduce the problem of k variables to a problem of $(k-1)$ variables, then we may have a chance to solve it.*

6.3 Resultant Method

In this section, we will develop a method that can reduce the above k-variate polynomial system to a $(k-1)$-variate polynomial system. We start with the case of $k = 2$. That is, we are given $2M + 2$ sampling values: $\{f_1, f_2, \ldots, f_{2M+2}\}$, in which two of them are incorrect. These sampling values satisfy the following equation

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f_{M+1} & f_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} & f_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f_{M+1} & f_{M+2} & f_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f_{M+2} & f_{M+3} & f_{M+4} & \cdots & f_{2M+1} & f_{2M+2} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \\ f_2 & f_4 & f_6 & \cdots & f_{2M} & f_{2M+2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \\ 1 \end{bmatrix} = 0, \tag{6.3.1}$$

where $p_1, \ldots, p_M$ are the coefficients of the Prony polynomial to be determined.

Since 2 of the $2M + 2$ sampling values are incorrect, we need to consider total $\binom{2M+2}{2}$ possible cases that two of the sampling values, say f_s and f_t for $1 \le s < t \le 2M + 2$, are incorrect. In order to make our discussion a little easier, we describe our method using a special case that $s = M + 1$ and $t = M + 2$ for visualization purpose. But our method can be applied on any general case for f_s and f_t.

Now we rewrite (6.3.1) with f^*_{M+1} and f^*_{M+2} representing the two corrupted sampling values, that is,

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f^*_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f^*_{M+1} & f^*_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f^*_{M+2} & f_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f^*_{M+1} & f^*_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f^*_{M+1} & f^*_{M+2} & f_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f^*_{M+2} & f_{M+3} & f_{M+4} & \cdots & f_{2M+1} & f_{2M+2} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \\ f_2 & f_4 & f_6 & \cdots & f_{2M} & f_{2M+2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \\ 1 \end{bmatrix} = 0. \tag{6.3.2}$$

We use Λ to denote the $(M+4)\times(M+1)$ matrix in (6.3.2), that is,

$$\Lambda := \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f^*_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f^*_{M+1} & f^*_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f^*_{M+2} & f_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f^*_{M+1} & f^*_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f^*_{M+1} & f^*_{M+2} & f_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f^*_{M+2} & f_{M+3} & f_{M+4} & \cdots & f_{2M+1} & f_{2M+2} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \\ f_2 & f_4 & f_6 & \cdots & f_{2M} & f_{2M+2} \end{bmatrix}. \tag{6.3.3}$$

(6.3.2) implies that Λ is not of full rank. Because if Λ is of full rank, then $\Lambda^T\Lambda$ is invertible, and we can get the following equation from (6.3.2)

$$\Lambda^T\Lambda \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \\ 1 \end{bmatrix} = 0. \tag{6.3.4}$$

This homogeneous equation (6.3.4) can only have the trivial solution, but the vector $[p_1, \ldots, p_M, 1]^T$ can never be zero, which is a contradiction. Thus, $\Lambda^T\Lambda$ is not invertible, and we have

$$\det(\Lambda^T\Lambda) = 0 \tag{6.3.5}$$

for the correct values for f_s and f_t, but it is not the zero polynomial.

On the other hand, $\Lambda^T\Lambda$ is a positive semi-definite matrix. Thus we always have

$$\det(\Lambda^T\Lambda) \geq 0. \tag{6.3.6}$$

Notice that $\Lambda^T\Lambda$ is a bivariate polynomial with respect to two variables: f_s and f_t. Let us use $F(f_s, f_t)$ to denote this polynomial,

$$F(f_s, f_t) := \det(\Lambda^T\Lambda). \tag{6.3.7}$$

Then (6.3.6) becomes

$$F(f_s, f_t) \geq 0 \qquad \text{for all } f_s, f_t \in \mathbb{R}. \tag{6.3.8}$$

Assume that f_s^0 and f_t^0 are the correct values for f_s and f_t, respectively. Then $F(f_s^0, f_t)$ is a univariate polynomial with respect to the variable f_t, and

$$F(f_s^0, f_t) \geq 0 \quad \text{for all } f_t \in \mathbb{R} \quad \text{and} \quad F(f_s^0, f_t^0) = 0. \tag{6.3.9}$$

Thus $F(f_s^0, f_t)$ takes a minimum at $f_t = f_t^0$, which is also a local minimum. Moreover, $F(f_s^0, f_t)$ is a polynomial of even degree (otherwise, it would not have a minimum), and it can be written as

$$F(f_s^0, f_t) = (f_t - f_t^0)^2 G(f_t) \tag{6.3.10}$$

for some polynomial $G(f_t)$. (6.3.10) implies that the polynomials $F(f_s^0, f_t)$ and $F'(f_s^0, f_t)$ have a common zero at $f_t = f_t^0$. That is, the *resultant* of $F(f_s^0, f_t)$ and $F'(f_s^0, f_t)$ is zero. Hence we have

$$\mathrm{Res}[F(f_s^0, f_t), F'(f_s^0, f_t)] = 0. \tag{6.3.11}$$

Observe that $\mathrm{Res}[F(f_s^0, f_t), F'(f_s^0, f_t)]$ does not contain the variable f_t, and we can treat it as a polynomial of f_s^0. Next we find all the real roots of $\mathrm{Res}[F(f_s^0, f_t), F'(f_s^0, f_t)]$ as the possible correct values for f_s. Then for each f_s^0, we can find f_t^0 as a local minimum of $F(f_s^0, f_t)$. With this pair (f_s^0, f_t^0), we can verify if they correspond to the solution using (6.3.2) in a similar way as in section (6.2).

6.4 Repeated Resultant Method

In this section, we will generalize the method developed in the previous section to the general case for $k > 1$. Since the discussion of the general case is very complicated, here we will use the $k = 3$ case to explain the main idea of this method. We are given $2M + 3$ sampling values: $\{f_1, f_2, \ldots, f_{2M+3}\}$, in which three of them are incorrect. These sampling values satisfy the following equation

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f_{M+1} & f_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f_{M+2} & f_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f_{M+1} & f_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f_{M+1} & f_{M+2} & f_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f_{M+2} & f_{M+3} & f_{M+4} & \cdots & f_{2M+1} & f_{2M+2} \\ f_{M+3} & f_{M+4} & f_{M+5} & \cdots & f_{2M+2} & f_{2M+3} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \\ f_2 & f_4 & f_6 & \cdots & f_{2M} & f_{2M+2} \\ f_3 & f_5 & f_7 & \cdots & f_{2M+1} & f_{2M+3} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \\ 1 \end{bmatrix} = 0, \tag{6.4.1}$$

where $p_1, \ldots, p_M$ are the coefficients of the Prony polynomial.

We will use f_u, f_v, and f_w to denote the three incorrect sampling values with $1 \leq u < v < w \leq 2M+3$. Thus we have total $\binom{2M+3}{3}$ cases to be considered. For simplicity, we describe our method using a special case that $u = M+1, v = M+2$ and $w = M+3$ for visualization purpose. But our method can be applied on any general case for f_u, f_v and f_w.

Now we rewrite (6.4.1) with f^*_{M+1}, f^*_{M+2} and f^*_{M+3} representing the three corrupted sampling values, that is,

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f^*_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f^*_{M+1} & f^*_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f^*_{M+2} & f^*_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f^*_{M+1} & f^*_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f^*_{M+1} & f^*_{M+2} & f^*_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f^*_{M+2} & f^*_{M+3} & f_{M+4} & \cdots & f_{2M+1} & f_{2M+2} \\ f^*_{M+3} & f_{M+4} & f_{M+5} & \cdots & f_{2M+2} & f_{2M+3} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \\ f_2 & f_4 & f_6 & \cdots & f_{2M} & f_{2M+2} \\ f_3 & f_5 & f_7 & \cdots & f_{2M+1} & f_{2M+3} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{M-1} \\ p_M \\ 1 \end{bmatrix} = 0. \tag{6.4.2}$$

We define $\Lambda(f_u, f_v, f_w)$ as the $(M+6) \times (M+1)$ matrix in (6.4.2), that is,

$$\Lambda(f_u, f_v, f_w) := \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_M & f^*_{M+1} \\ f_2 & f_3 & f_4 & \cdots & f^*_{M+1} & f^*_{M+2} \\ f_3 & f_4 & f_5 & \cdots & f^*_{M+2} & f^*_{M+3} \\ & \vdots & & \cdots & & \vdots \\ f_M & f^*_{M+1} & f^*_{M+2} & \cdots & f_{2M-1} & f_{2M} \\ f^*_{M+1} & f^*_{M+2} & f^*_{M+3} & \cdots & f_{2M} & f_{2M+1} \\ f^*_{M+2} & f^*_{M+3} & f_{M+4} & \cdots & f_{2M+1} & f_{2M+2} \\ f^*_{M+3} & f_{M+4} & f_{M+5} & \cdots & f_{2M+2} & f_{2M+3} \\ f_1 & f_3 & f_5 & \cdots & f_{2M-1} & f_{2M+1} \\ f_2 & f_4 & f_6 & \cdots & f_{2M} & f_{2M+2} \\ f_3 & f_5 & f_7 & \cdots & f_{2M+1} & f_{2M+3} \end{bmatrix}. \tag{6.4.3}$$

Similar to the discussion in section (6.3), we have

$$\det\big((\Lambda^T\Lambda)(f_u, f_v, f_w)\big) \geq 0 \qquad \text{for all } f_u, f_v, f_w \in \mathbb{R}. \tag{6.4.4}$$

In particular, when we take the correct values for f_u, f_v, and f_w, say $f_u = f^0_u, f_v = f^0_v$, and $f_w = f^0_w$, we get

$$\det\big((\Lambda^T\Lambda)(f^0_u, f^0_v, f^0_w)\big) = 0. \tag{6.4.5}$$

Define the trivariate polynomial $F(f_u, f_v, f_w)$ as,

$$F(f_u, f_v, f_w) := \det\left((\Lambda^T\Lambda)(f_u, f_v, f_w)\right). \tag{6.4.6}$$

Then we have

$$F(f_u, f_v, f_w) \geq 0 \qquad \text{for all } f_u, f_v, f_w \in \mathbb{R}, \tag{6.4.7}$$

and

$$F(f_u^0, f_v^0, f_w^0) = 0. \tag{6.4.8}$$

Now we consider the univariate polynomial $F(f_u^0, f_v^0, f_w)$ with respect to the variable f_w. (6.4.7) and (6.4.8) imply that

$$F(f_u^0, f_v^0, f_w) \geq 0 \quad \text{for all } f_w \in \mathbb{R} \quad \text{and} \quad F(f_u^0, f_v^0, f_w^0) = 0, \tag{6.4.9}$$

which means that $F(f_u^0, f_v^0, f_w)$ takes a (local) minimum at $f_w = f_w^0$, and we can write

$$F(f_u^0, f_v^0, f_w) = (f_w - f_w^0)^2 G(f_w) \tag{6.4.10}$$

for some polynomial $G(f_w) \geq 0$ for all $f_w \in \mathbb{R}$. (6.4.10) implies that the polynomials $F(f_u^0, f_v^0, f_w)$ and its derivative $F'(f_u^0, f_v^0, f_w)$ have a common zero at $f_w = f_w^0$. That is, the *resultant* of $F(f_u^0, f_v^0, f_w)$ and $F'(f_u^0, f_v^0, f_w)$ is zero. Hence we have

$$\mathrm{Res}[F(f_u^0, f_v^0, f_w), F'(f_u^0, f_v^0, f_w)] = 0. \tag{6.4.11}$$

Observe that $\mathrm{Res}[F(f_u^0, f_v^0, f_w), F'(f_u^0, f_v^0, f_w)]$ does not contain the variable f_w, thus it is a bivariate polynomial with respect to f_u^0 and f_v^0.

Now we consider the Sylvester matrix for $F(f_u^0, f_v^0, f_w)$ and $F'(f_u^0, f_v^0, f_w)$, and use the following notations

$$\Lambda_1(f_u^0, f_v^0) := \mathrm{Syl}[F(f_u^0, f_v^0, f_w), F'(f_u^0, f_v^0, f_w)] \tag{6.4.12}$$

and

$$F_1(f_u, f_v) := \det\left((\Lambda_1^T\Lambda_1)(f_u, f_v)\right). \tag{6.4.13}$$

It is easy to see that

$$F_1(f_u, f_v) \geq 0 \quad \text{for all } f_u, f_v \in \mathbb{R} \quad \text{and} \quad F_1(f_u^0, f_v^0) = 0. \tag{6.4.14}$$

Now we consider the univariate polynomial $F_1(f_u^0, f_v)$ with respect to the variable f_v, and from (6.4.14) we get

$$F_1(f_u^0, f_v) \geq 0 \quad \text{for all } f_v \in \mathbb{R} \quad \text{and} \quad F_1(f_u^0, f_v^0) = 0, \tag{6.4.15}$$

which means that $F_1(f_u^0, f_v)$ takes a (local) minimum at $f_v = f_v^0$, and we can write

$$F_1(f_u^0, f_v) = (f_v - f_v^0)^2 G_1(f_v) \tag{6.4.16}$$

for some polynomial $G_1(f_v) \geq 0$ for all $f_v \in \mathbb{R}$. The structure of (6.4.16) implies that

$$\mathrm{Res}[F_1(f_u^0, f_v), F_1'(f_u^0, f_v)] = 0. \tag{6.4.17}$$

Since $\mathrm{Res}[F_1(f_u^0, f_v), F_1'(f_u^0, f_v)]$ is a univariate polynomial for variable f_u^0, we can find all the real zeros of $\mathrm{Res}[F_1(f_u^0, f_v), F_1'(f_u^0, f_v)]$, denoted by a set Z_0, as the possible correct values for f_u.

Next we select a $f_u^0 \in Z_0$, and with this f_u^0, we try to find f_v^0 that satisfies (6.4.16). This can be done by looking at all the zeros of $F_1(f_u^0, f_v)$ with the property that all the real zeros have even multiplicities and the leading coefficient is positive. If we are successful at this step, we will use this pair of f_u^0 and f_v^0 to find f_w^0, such that (6.4.10) is true by examining the multiplicities of the real zeros of $F(f_u^0, f_v^0, f_w)$ and its leading coefficient. If we are successful at this step, we find a triple (f_u^0, f_v^0, f_w^0) that could be the correct values of f_u, f_v, and f_w.

If we find multiple triples of (f_u^0, f_v^0, f_w^0)'s that satisfy (6.4.8), then we need to go to the verification step to make sure that they are the correct values. The verification step can be implemented as follows:

- Use the triple (f_u^0, f_v^0, f_w^0) to find $p_1, p_2, \ldots, p_M$ that satisfy (6.4.1). If this step is successful, we move to the next step; otherwise, we move to the next triple.

- With these $p_1, p_2, \ldots, p_M$, we can recover the expansion (6.1.1), and get the function $f(x)$.

- With this $f(x)$, we can evaluate the sampling values $f(nh), n = 1, 2, \ldots, 2M+3$, then compare them with the given sampling values and see if they are compatible. If they are compatible, we find a solution; otherwise, we move to the next triple and restart the verification process.

Chapter 7

Summary and Future Work

In this chapter, we make the conclusion of this dissertation and describe our future research problems. We have studied several cases in data analysis to recover the modelling functions in the sparse expansion expressions from equispaced sampling values using the original Prony method and some of its generalizations.

In some expansions, we are able to use the frequency domain to get the sampling values. In this case, we transform the modelling functions from the time domain to the frequency domain through the short time Fourier transform (STFT). This approach is very successful in image processing, because the coefficient modification in the frequency domain does not have a sensitive consequence in the time domain, which is also desirable in data analysis. This method works only for certain special basis functions, such as the B-splines, the exponential functions, and the cosine functions.

The generalized Prony method was very helpful in this dissertation. It allows us to recover the functions that are expansions of the eigenfunctions of some linear operators. Moreover, we could also reconstruct some expansions that have two different models using the generalized Prony method.

There are several research problems related to this topic. Other than the problems we studied in this dissertation, we have the following list of problems for our future research:

1. (*Apply various transforms on the basis functions*)

   Other than the Fourier transform and the short time Fourier transform, we will explore several other transforms, such as the Laplace transform, so that we can expand our processing power to cover more basis functions in the frequency domain. Working in the frequency domain allows us to avoid the sensitivity of the Prony method in the time domain.

2. (*Discover more operators with useful eigenfunctions*)

   Some basis functions that cannot be handled by certain transform can still be processed by the Prony method if they happen to be the eigenfunctions of a linear operator. If we can discover more such operators, we can enlarge the coverage of the Prony method for more useful basis functions.

3. (*Apply the Prony method on splines with multiple knots*)

   Currently we can only apply the Prony method on splines with single knots. In many real-world applications, we need to use splines with multiple knots, such as the Hermit interpolations. The original Prony method requires that all the frequency parameters must be distinct, which is not compatible with the multiple knots for the splines. A new technique should be developed to handle the structure caused by the multiple knots.

4. (*Recover the signals containing corrupted sampling values*)

   In Chapter 6, we solved the problem with certain number of incorrect sampling values. Since the whole process requires a lot of computation, the efficiency of the algorithm is very important. Is there any way to help us reduce the number of tries?

   Problem A2: To recover a sparse expansion system with M terms as follows,

   $$f(x) = \sum_{j=1}^{M} c_j e^{\phi_j x},$$

   we are given a sequence of *equidistance* sampling values $S := \{f(nh)\}_{n=1}^{2M+k}$ for some fixed step size h. It is also known that there are exact k sample values in S that are incorrect, but we do not know their locations. How to recover this system?

   In Chapter 6, we develop a method that solves this problem, but each time we need to assume k sampling values that are incorrect for one experiment. The total number of experiments we need to do could be as high as $\binom{2M+k}{k}$, which could be a very large number. To reduce the signal recovery cost, we need some detection method that can help us eliminate a large number of possibilities. In other words, we need to develop some method that can tell us that some of the sampling values are incorrect. With that kind of partial certainty, the number of possibilities that we need to try can be greatly reduced.

5. (*Recover the signals containing corrupted sampling values without oversampling*)

   In some real-world applications, we may not have oversampling values to overcome certain number of incorrect sampling values, but we still need to recover the signals. Is it possible?

   The method we developed in Chapter 6 has the potential to solve this problem. For example, we are given $2M$ equispaced sampling values, among which there are k (a relatively small number) incorrect values. Can we still recover the original signals? We will investigate if our *Repeated Resultant* method can solve this problem.

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